VIBRATION ANALYSIS OF A SIMPLY SUPPORTED CYLINDRICAL SHELL

A thesis submitted to the department of Mechanical and Chemical Engineering (MCE), Islamic University of Technology (IUT), in the partial fulfillment of the requirement for the degree of Bachelor of Science in Mechanical Engineering.

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Declaration

This is to certify that the work presented in this thesis is an outcome of the experiment and research carried out by the authors under the supervision of Dr. Zahid Hossain and Dr. Mir Md. Maruf Morshed.

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Date: Date:

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ABSTRACT

The linear response of a cylindrical shell (thin cylinder) subjected to modal and harmonic excitations are investigated. Natural frequencies and forced vibration response are investigated for the simply supported-simply supported boundary conditions. The equations of motion of the structure for the theoretical analysis are obtained from Love's equation and for computing results, programs are written in MATLAB. Finite element method is used for numerical analysis (using ANSYS MECHANICAL APDL). The natural frequencies obtained by numerical, theoretical and experimental analyses were compared and showed good agreement among the results.

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CHAPTER 1

INTRODUCTION

Thin walled cylindrical shells are very often in the field of engineering applications. Shell structures are common especially in spacecraft, aircraft, shipbuilding and automotive industries. They are also used as oil and gas carrying pipelines. Present day sees a great increase in the applications of cylindrical shells in the form of structural components for pressure vessels, process equipments, missiles, rockets and civil engineering constructions. Most of the failures occurring in these structures are due to dynamic loading. For instance, the infamous tragic failure of NASA space shuttle CHALLENGER on 20th January, 1986 was mainly due to structural failure. That's why vibration analysis of shell structures has been of great importance for last few decades.

The natural frequencies and mode shapes are important sources of information for understanding and controlling the vibration of these structures.

Such cylindrical structures are often subjected to dynamic loading. Such as the flow-induced vibrations in heat exchangers and pipelines, wave-loading on submarines, the impact-loading of vehicles, the aero-elastic flutter of aircrafts, vibrations of underground and under-sea pipe- lines and certain defense-related equipment. Vibrations in the aforementioned pipelines are generally caused by external driving agencies such as earthquakes, nuclear and other explosions, wave-loadings, superfast trains and super-sonic jets. Thin-walled structures are very prone to resonant vibrations because their eigenfrequencies lie in a very narrow band. So it is essential for the designer to know the distribution of eigenfrequencies of the proposed structure beforehand.

Therefore main focus of our work was to find out the natural frequencies of a thin walled cylindrical shell (modal analysis) at different end conditions and observe its behavior under external loading conditions (harmonic analysis).

This analysis consists of three phases namely Numerical Analysis, Analytical analysis and experimental analysis. In numerical analysis ANSYS (Mechanical APDL) was used to find the natural frequencies and extract the mode shapes and to observe the response under external loading (Harmonic analysis). In analytic analysis the Love's approach was used to find the equations of motions and then the eigenvalues were calculated. Then analytically harmonic analysis was carried

out. For experimental analysis an inductive proximity sensor was used to find natural frequency of the shell.

LITERATURE REVIEW

As we have already known the importance of shell structure analysis from the introduction before, now we'll look into various shell theories those have paved the way to the present day extensive shell analysis.

Regarding researches of shell vibration, Leissa [1] has collected most of the results before 1973. Chung [2] and Greif and Chung [3] used the Rayleigh-Ritz method, for different boundary conditions, to find the natural frequencies. Sharma and Johns [4, 5] and Goldman [6] calculated the natural frequencies and modes for free and fixed boundary conditions. Stoke's transformation technique was applied by Chung [7] to solve the natural frequencies for different boundary conditions. Mnev and Pertsev [8], Junger and Feit [9], and Brown [10] have done some research on vibration of shells with the interaction of internal fluid. Chu et el. [11] used the energy method to obtain the frequency parameters. Recent works include Goncalves' [12] investigation of non-linear vibrations of thin-walled cylinders with liquid interaction.

Markus [13] has provided an extensive analysis of cylindrical shells using membrane as well as bending theory. He has discussed the cons and pros of the membrane theory. He discussed various shell theories due to Donnell-Mushtari, Love-Timoshenko, FlÜgge, Sander etc.

In recent years Bert et al. [14] have given an analytical solution to the free vibration of a composite material cylindrical shell with ring and stringer stiffeners and compared the numerical values given by various shell theories, by the use of dimensionless tracer coefficients. Mustafa and Ali [15-17] have predicted natural frequencies of stringer stiffened and ring stiffened cylindrical shells using semi-loof and facet shell finite elements on half and quarter models of the shells, because of structural symmetry. They have compared the numerical values obtained by them, with the experimental values of Hoppmann [18].They have also given an energy method to study the natural frequencies of externally and internally stinger stiffened cylindrical shells and ring stiffened shells. Rinehart and Wang [19, 20] have investigated the free vibration characteristics of Simply-supported cylindrical shells stiffened by discrete longitudinal stiffeners using energy method. They have compared the numerical values given by the more exact F'lugge's theory and Donnell's approximate theory and shown that Donnell's approximate theory gives excellent results for the stiffened shells.

Previous studies confirmed that the effect of shear deformation can become quite significant for small radius-to- thickness or length-to-thickness ratios, as well as for shorter wavelengths of longer shells [21]. More recently, Bhimaraddi [21] developed a two-dimensional (2-D) higher-order shell theory for free vibration response of isotropic circular cylindrical shell and assumed the inner and outer surfaces of the shell to be traction free. Also, Reddy and Liu [22] presented a 2-D higher-order theory for laminated elastic shells.

CHAPTER 2

BASICS OF VIBRATION

TYPES OF VIBRATION

FREE AND FORCED VIBRATION

FREE VIBRATION

After an initial disturbance, if a system is left to vibrate on its own then it is called free vibration. In free vibration no external force is applied or acted on the system. Oscillation of a simple pendulum is an example of free vibration.

FORCED VIBRATION

If a system is subjected to an external force (often repeating types) the resulting vibration is known as forced vibration. The oscillation that arises in machine such as diesel engine is an example of forced vibration.

UNDAMPED AND DAMPED VIBRATION

UNDAMPED VIBRATION

During oscillation if no energy is lost or dissipated due to friction or other resistances then the vibration is known as **undamped vibration**. In an undamped vibration the magnitude of amplitude is not changing with time.
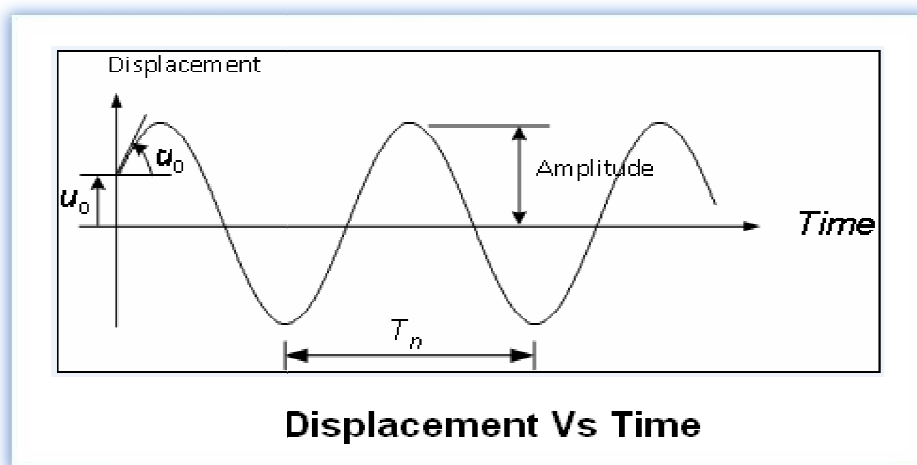
Figure: undamped vibration

DAMPED VIBRATION:

During oscillation if energy is lost due to friction or other resistances then it is called **damped vibration**. During damped vibration the magnitude of amplitude or displacement is changed with time.
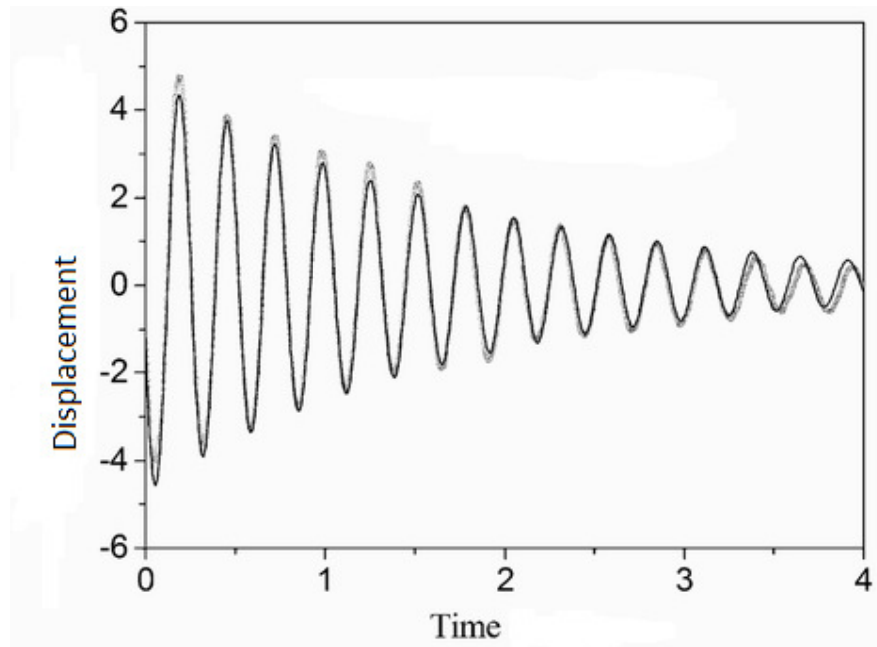
Figure: Damped vibration

LINEAR VIBRATION AND NONLINEAR VIBRATION:

LINEAR VIBRATION:

If all basic components of a vibratory system –the spring, the mass, and the damper- behave linearly, the resulting vibration is known as the **linear vibration**. If the vibration is linear then the principle of superposition holds.

Nonlinear vibration:

If any of the basic components of vibration behave nonlinearly then the vibration is called **nonlinear vibration**. For nonlinear vibration the principle of superposition is not valid.

STEADY STATE VIBRATION OR RESPONSE

At forced vibration the system will tend to vibrate at its own natural frequency and to follow the frequency of the external force applied. In the presence of friction the portion of motion not sustained by the excitation force will gradually die out. In other words due to friction the tendency of vibrating at natural frequency will be eliminated. As a result the system will vibrate at the frequency of external force only regardless of the initial conditions or the natural frequency of the system. This part of sustained vibration is called the **steady state vibration or response** of the system. Very often the steady state response is required in vibration analysis because of its continuous effect.
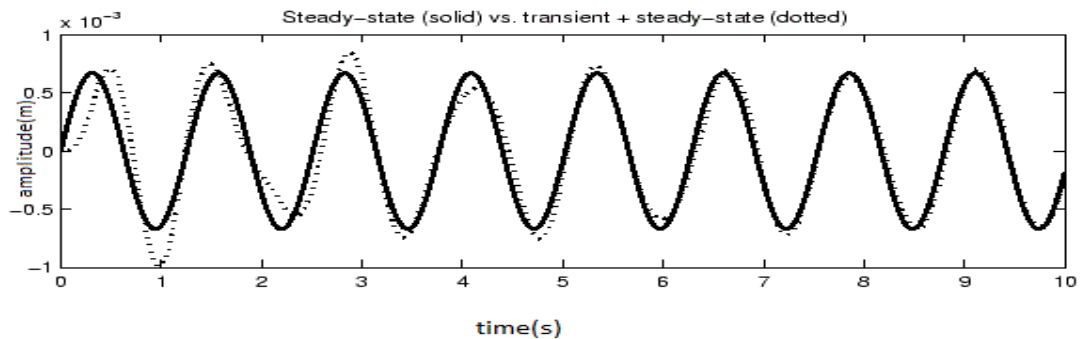
Figure: steady state vibration

THIN –WALLED CYLINDER

For the thin-walled assumption to be valid the vessel must have a wall thickness of no more than about one-tenth (often cited as one twentieth) of its radius. This allows for treating the wall as a surface, and subsequently using the **Young–Laplace equation** for estimating the hoop stress created by an internal pressure on a thin wall cylindrical pressure vessel:

$$\sigma_\theta = \frac{Pr}{t} \quad \text{(For a cylinder)}$$

$$\sigma_\theta = \frac{Pr}{2t} \text{(For a sphere)}$$

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Where,

- P is the internal pressure
- t is the wall thickness
- r is the inside radius of the cylinder.
- σ_θ is the hoop stress.

THICK-WALLED CYLINDER

When the cylinder to be studied has a r/t ratio of less than 10 (often cited as 20) the thin-walled cylinder equations no longer hold since stresses vary significantly between inside and outside surfaces and shear stress through the cross section can no longer be neglected.

In order to calculate the stresses and strains here a set of equations known as the **Lamé** equations must be used.

$$\sigma_r = A - \frac{B}{r^2}$$

$$\sigma_\theta = A - \frac{B}{r^2}$$

Where,

- A and B are constants of integration, which may be discovered from the boundary conditions
- r is the radius at the point of interest (e.g., at the inside or outside walls)

A and B may be found by inspection of the boundary conditions.

RESONANCE

A certain system has more than one natural frequency. If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as resonance occurs. When resonance happens, the amplitude of vibration will increase without bound and is governed only by the amount of damping present in the system and the system undergoes dangerously large oscillations. Therefore, in order to avoid disastrous effects resulting from very large amplitude of vibration at resonance the natural frequency of a system must be known and properly taken care of. Otherwise failures of such structures as buildings, bridges, turbines and airplane wings may be occurred.
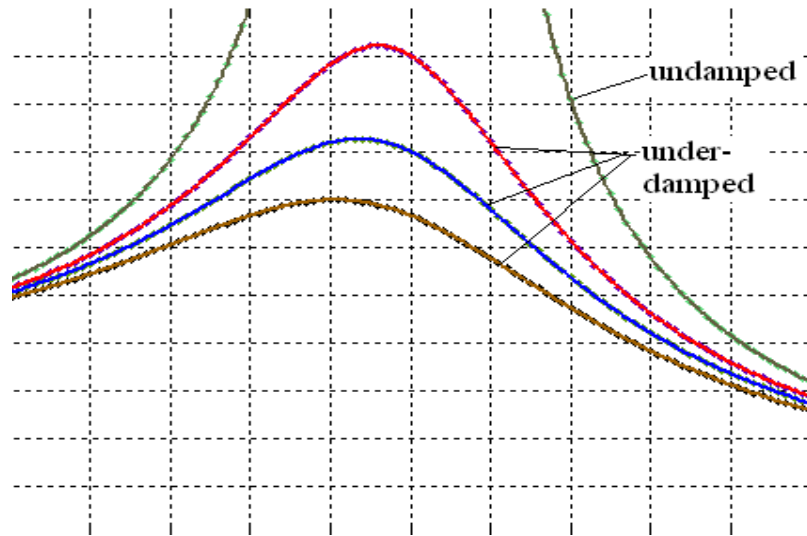
Figure: resonance curve

MODAL ANALYSIS:

Modal analysis is the study of the dynamic properties of structures under vibrational excitation.

Modal analysis is the field of measuring and analyzing the dynamic response of structures and or fluids when excited by an input. Examples would include measuring the vibration of a car's body when it is attached to an electromagnetic shaker, or the Modal analysis is the study of the dynamic properties of structures under vibrational excitation.

The goal of modal analysis in structural mechanics is to determine the natural mode shapes and frequencies of an object or structure during free vibration. It is common to use the finite element method (FEM) to perform this analysis because, like other calculations using the FEM, the object being analyzed can have arbitrary shape and the results of the calculations are acceptable. The types of equations which arise from modal analysis are those seen in Eigen systems. The physical interpretation of the eigenvalues and eigenvectors which come from solving the system are that they represent the frequencies and corresponding mode shapes. Sometimes, the only desired modes are the lowest frequencies because they can be the most prominent modes at which the object will vibrate, dominating all the higher frequency modes.

HARMONIC ANALYSIS

Any sustained cyclic load will produce a sustained cyclic response (a harmonic response) in a structural system. Harmonic response analysis gives the ability to predict the sustained dynamic behavior of your structures, thus enabling you to verify whether or not your designs will successfully overcome resonance, fatigue, and other harmful effects of forced vibrations.

Harmonic response analysis is a technique used to determine the steady-state response of a linear structure to loads that vary sinusoidally (harmonically) with time. The idea is to calculate the structure's response at several frequencies and obtain a graph of some response quantity (usually displacements) versus frequency. "Peak" responses are then identified on the graph and stresses reviewed at those peak frequencies.

This analysis technique calculates only the steady-state, forced vibrations of a structure. The transient vibrations, which occur at the beginning of the excitation, are not accounted for in a harmonic response analysis.

CHAPTER 3

NUMERICAL ANALYSIS FOR DETERMINING NATURAL FREQUENCIES AND FORCED VIBRATION RESPONSE OF A CIRCULAR CYLINDRICAL SHELL

NUMERICAL ANALYSIS

Numerical analysis is the study of algorithms or step by step process that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis (as distinguished from discrete mathematics).

A numerical method which leads to a required result is often referred to as an algorithm. More often than not, algorithms are iterative, i.e., they involve cycles of identical computations, starting with the results of the preceding cycle. At the end of a cycle, the result will be examined to find out whether it has the required accuracy. The algorithm will stop, when the error becomes as small as desired.

USE OF ANSYS IN MODAL ANALYSIS

Modal analysis is generally used to determine the natural frequencies of different structures. That later on serves as the starting point of another more detailed, dynamic analysis such as transient harmonic analysis or a harmonic response analysis.

Modal analysis in ANSYS family uses linear analysis method. Any sorts of nonlinearities such as plasticity and contact (gap) elements are ignored even if they are defined. We use ANSYS in finding natural frequencies of any structure and extract mode shapes. One can choose from several mode extraction methods: Block Lanczos, Supernode, PCG Lanczos, reduced, unsymmetric, damped and QR damped.

COMPARING MODE-EXTRACTION METHODS

The basic equation solved in a typical undamped modal analysis is the classical eigenvalue problem:

$$[K]\{\phi_i\} = \omega_i^2[M]\{\phi_i\}$$

Where,

$[K]$= stiffness matrix

$\{\phi_i\}$= mode shape vector (eigenvector) of mode i

Ω_i= natural circular frequency of mode i (ω_i^2 is the eigenvalue)

$[M]$= mass matrix
Many numerical methods are available to solve the above equation. ANSYS offers these methods:
• Block Lanczos method
• PCG Lanczos method
• Supernode (SNODE) method
• Reduced (Householder) method
• Unsymmetric method
• Damped method (The damped method solves a different equation)

PROCESSES INVOLVED IN MODAL ANALYSIS IN ANSYS

The general process for a modal analysis involves following primary steps:

1. Building the model.
2. Apply the loads and obtain the solution.
3. Expand the modes.
4. Review the results.

ANSYS SIMULATION OF A CYLINDRICAL SHELL

Now the process described above will be applied for a cylindrical shell to analyze its natural frequencies and expand the mode shapes. One thing should be mentioned over here that the simulation can be performed two ways: by coding and by using the GUI (Graphical User Interface). We would be using here the later one.

BUILDING THE MODEL IN ANSYS MECHANICAL (APDL)

Firstly, the shell should be modeled.
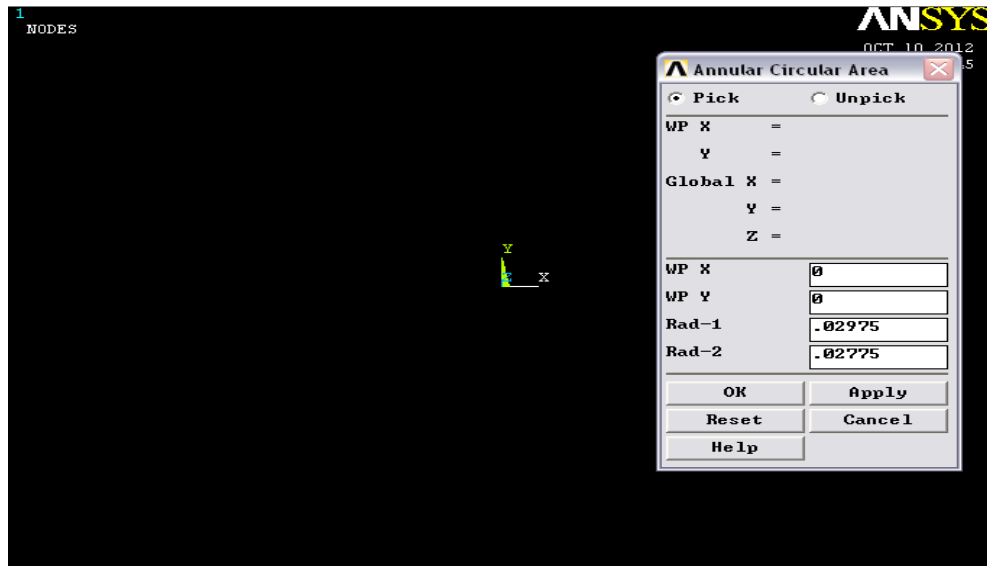
Entering work plane co-ordinates as (0, 0) we specified radiuses to be **0.02975** & **0.02775**.One thing should kept in mind that all the units should be kept under the same system throughout the whole simulation process. The annular circle should look like below.
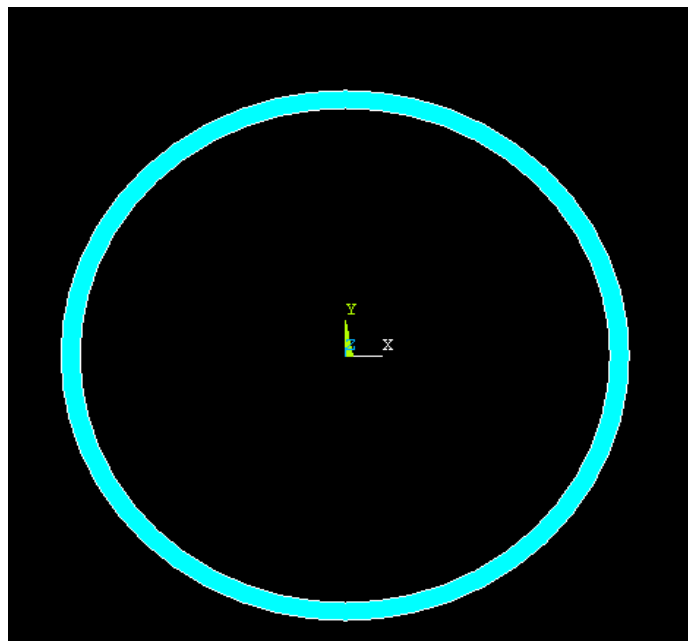
Secondly, the type of simulation to be done is to be selected. Steps involved:
Opening the GUI>Preferences>Structural
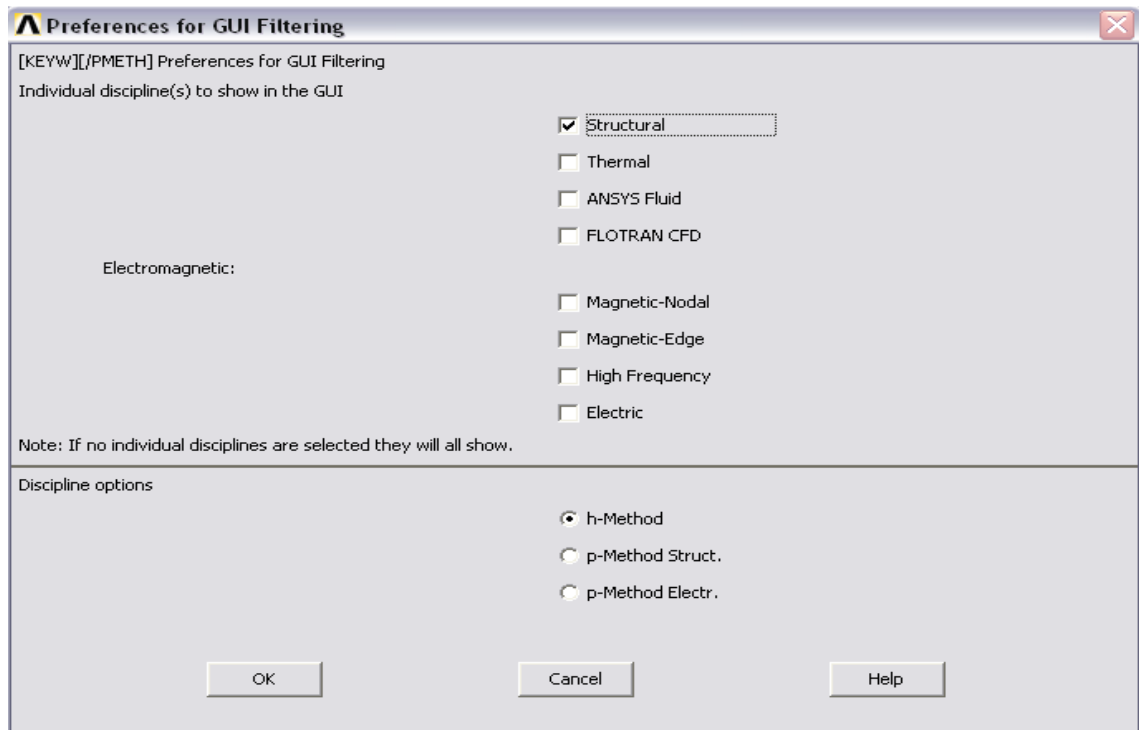
Thirdly, constant material properties (such as Young's modulus of elasticity, poison's ratio and density) are to be defined. **Main menu** > **Preprocessor**> **Material Props** > **Material Models**.
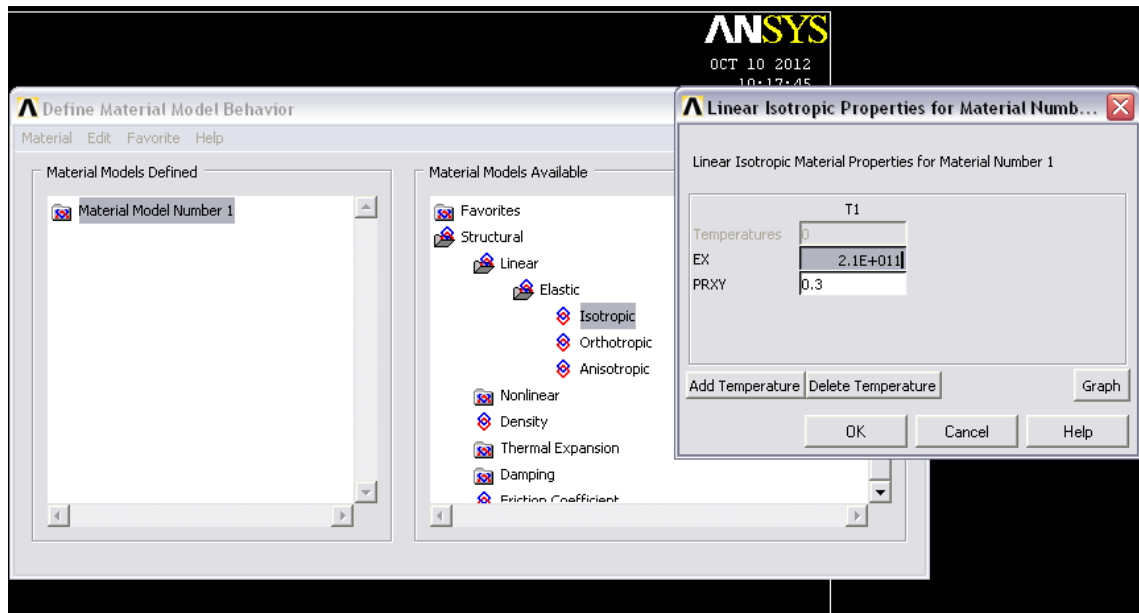
For density,
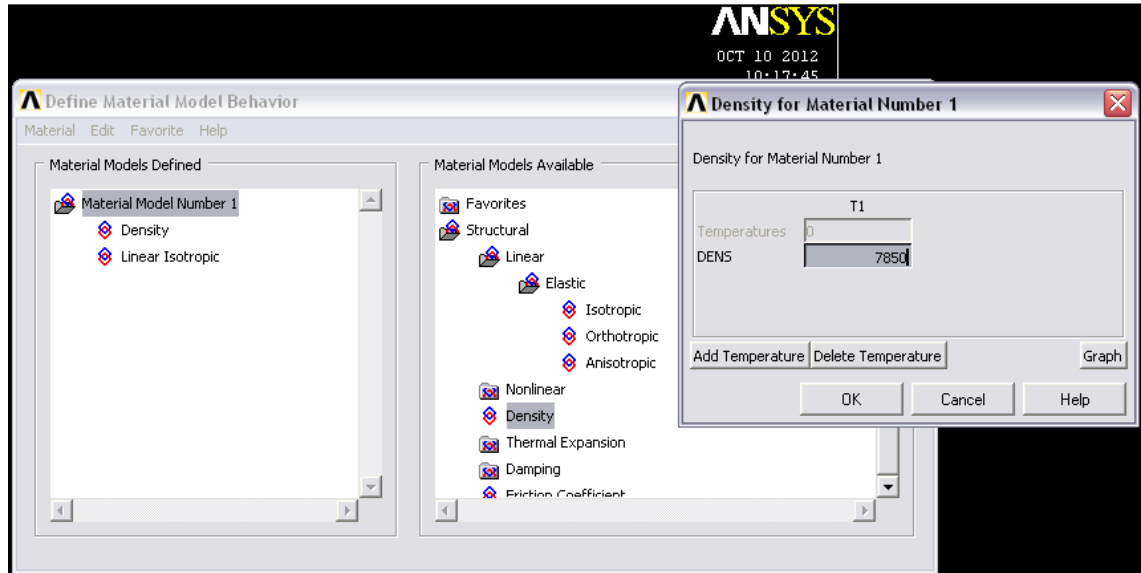
Then we have to generate the mesh. To do that we need to define the element type. The meshing process can be told in a nutshell this way: at first the 2D element PLANE 42 is added then the 3D element SOLID 45 is added. The 2D element is meshed then extruded using 3D element. Thus whole object gets extruded.

Main Menu>Preprocessor>Element Type>Add/Edit/Delete.

For selecting 2D **PLANE 42**:
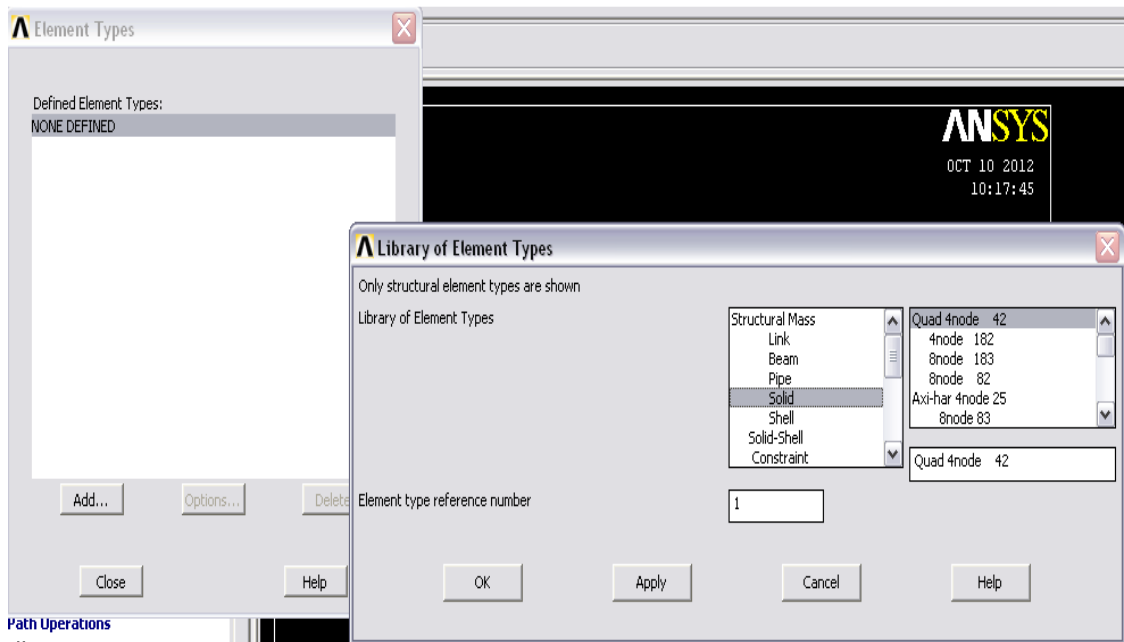
Then for selecting 3D element **SOLID 45**:
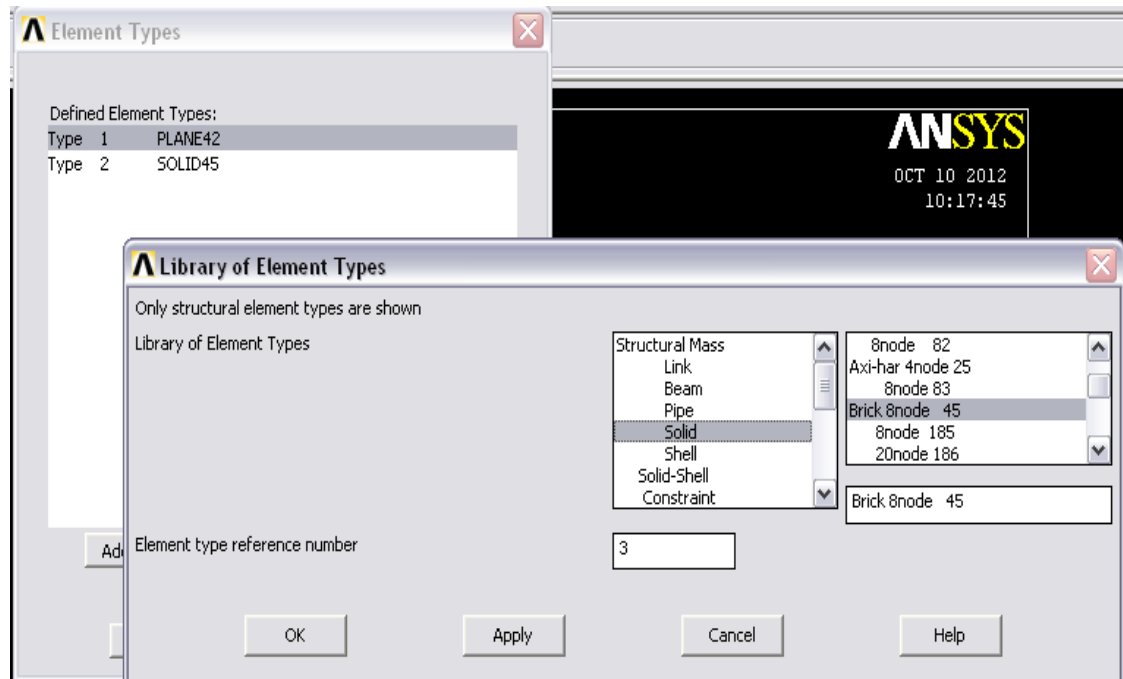
PLANE 42

PLANE42 is used for 2-D modeling of solid structures. The element can be used either as a plane element (plane stress or plane strain) or as an axisymmetric element. The element is defined by four nodes having two degrees of freedom at each node: translations in the nodal x and y directions. The element has plasticity, creep, swelling, stress stiffening, large deflection, and large strain capabilities.
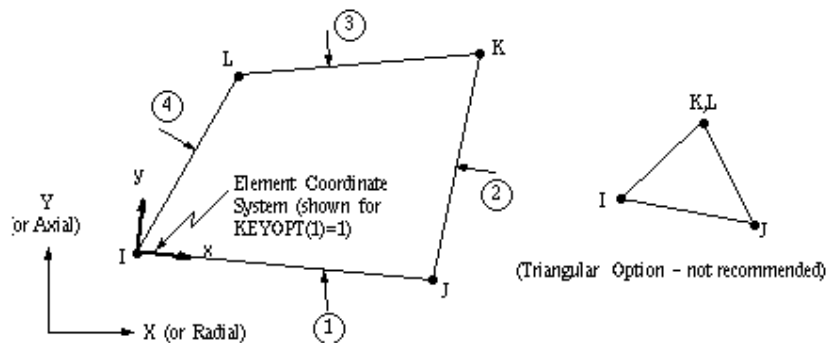
Fig: PLANE 42, Structural element

SOLID 45

SOLID45 is used for the three-dimensional modeling of solid structures. The element is defined by eight nodes having three degrees of freedom at each node: translations in the nodal x, y, and z directions.

The element has plasticity, creep, swelling, stress stiffening, large deflection, and large strain capabilities. A reduced integration option with hourglass control is available.
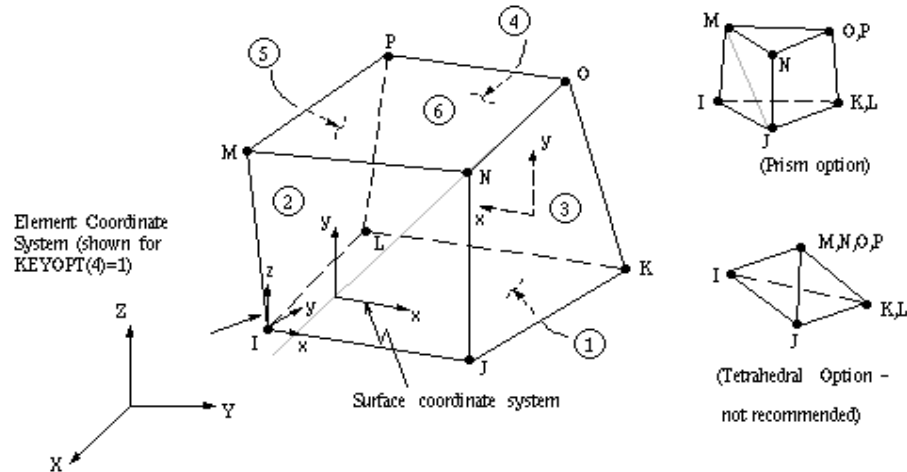
Fig:SOLID 45, Structural element.

Now the 2D element is meshed by specifying through mesh controls. **Main Menu>Preprocessor>Meshing>Mesh Tool**
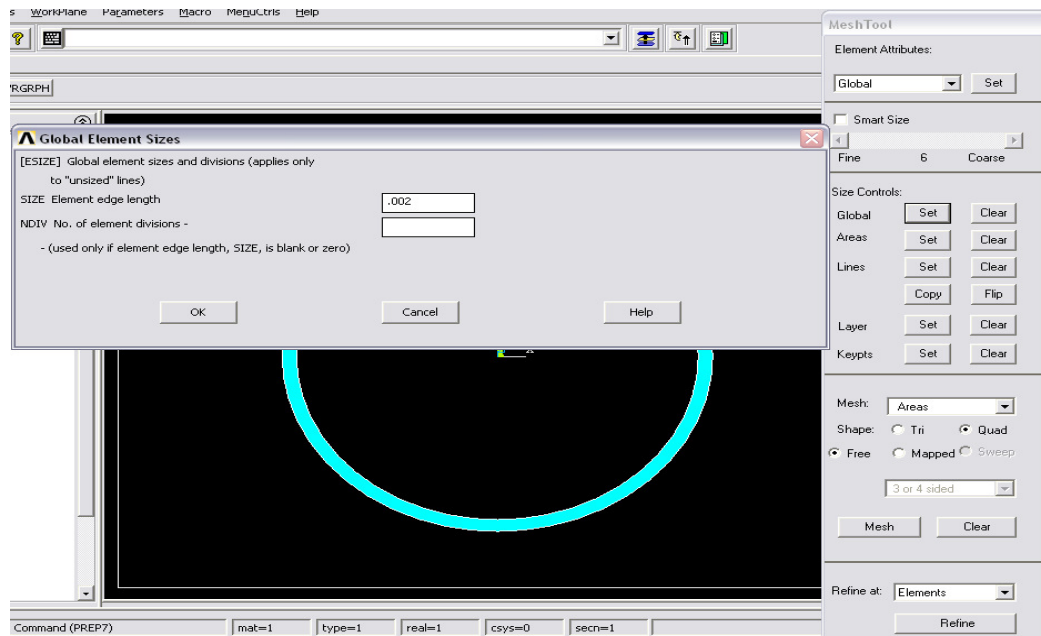
Here for meshing purpose we have taken:

Edge length=0.002m. That is each small element has a side of .002m. **Main Menu> Preprocessor> Meshing> Mesh Tool**

Step by step process:

1. "**Size Controls Global**" = [Set]

2. "**Element edge length**" = 0.002

3. [OK]

4. [Mesh]

5. [Pick All]

6. [Close] Warning. After that the 2D element is extruded: **Main Menu> Preprocessor> Modeling> Operate> Extrude> Elem Ext Opts**
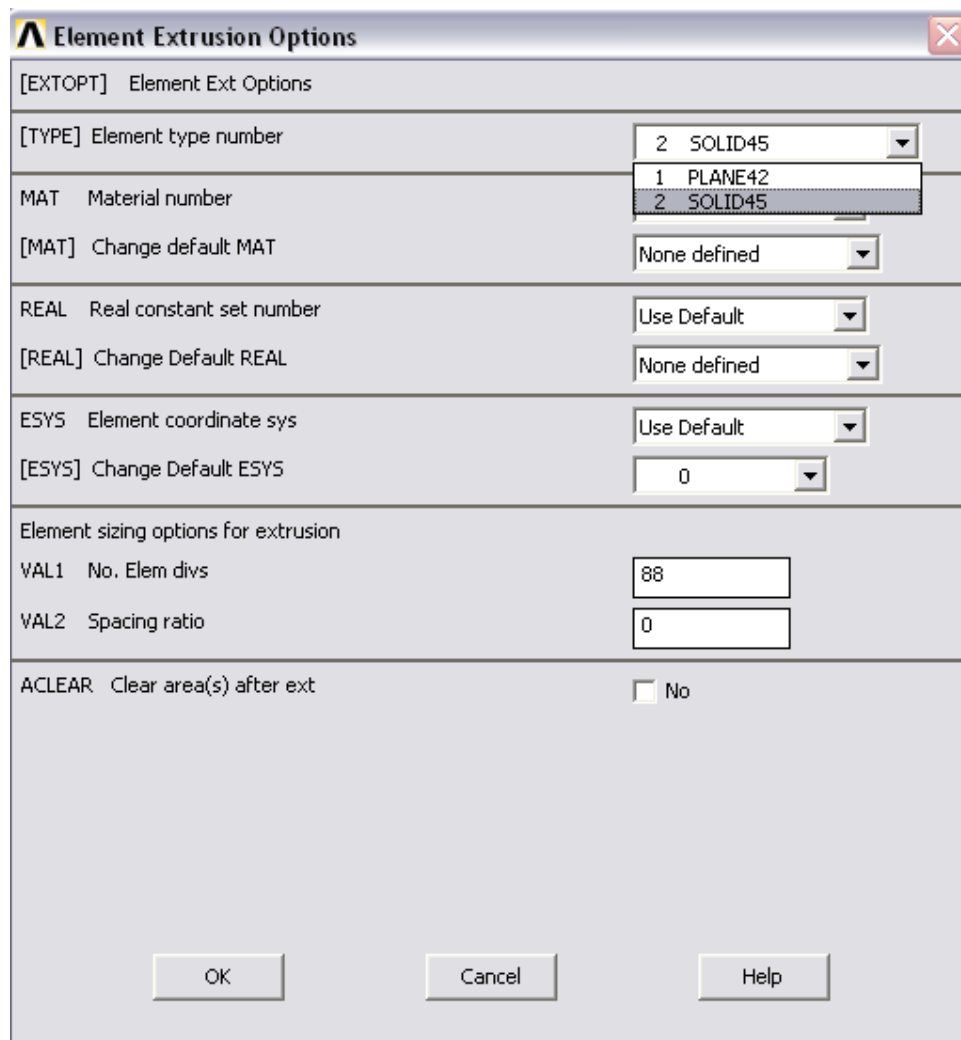
Element type number is changed to **SOLID 45** & **No. Elem divs =**88. Then the 2D element is extruded**: Main Menu> Preprocessor> Modeling> Operate> Extrude> Areas> By XYZ Offset> [Pick All].**

Here "**Offsets for extrusion**" = 0, 0, 2.2.
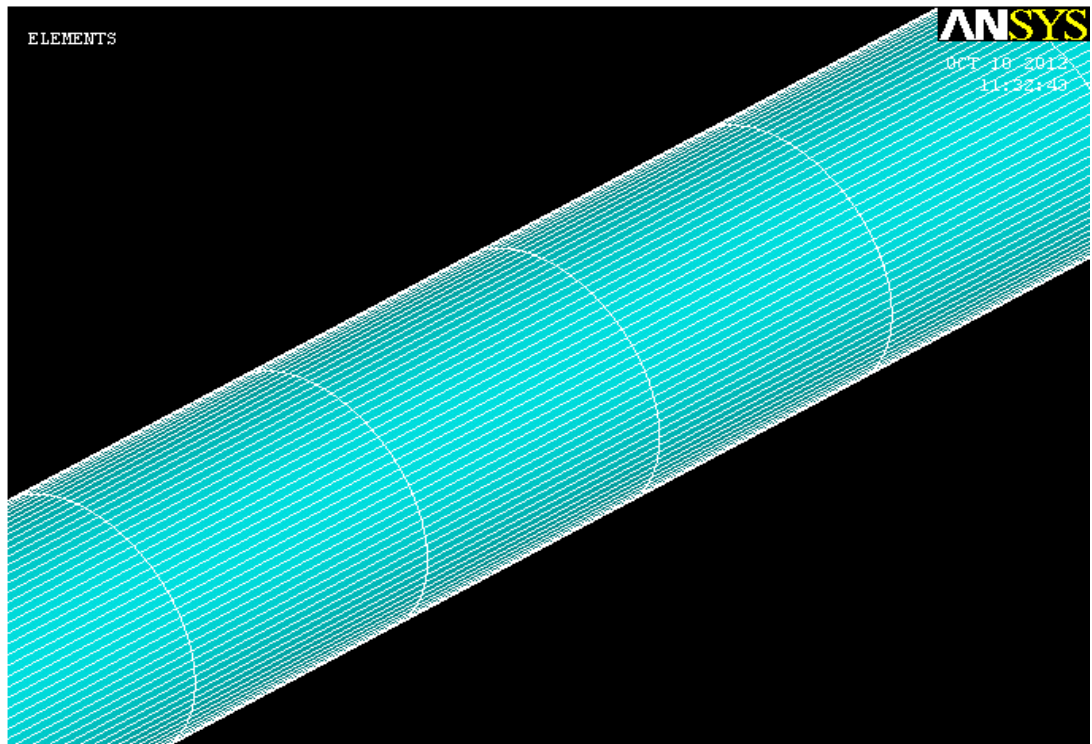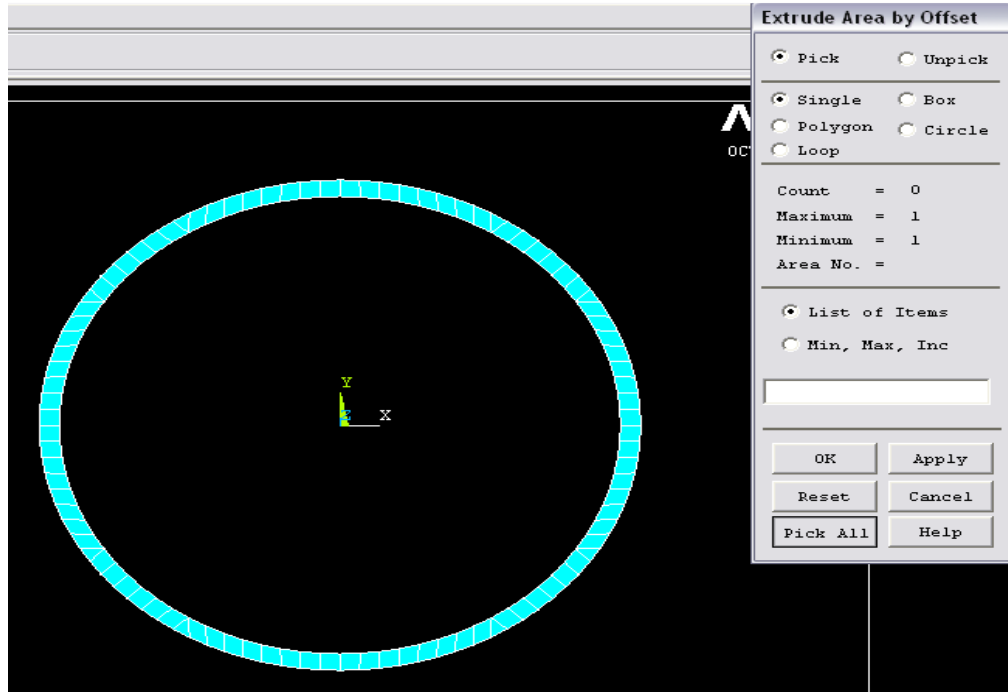
The final meshed model looks like the above figure (zoomed view).

APPLYING THE LOADS AND OBTAINING THE SOLUTION

This part can be described into several categories:

Step 1: Unselect 2-D elements.

Before applying constraints to the fixed end of the wing, unselect all **PLANE42** elements used in the 2-D area mesh since they will not be used for the analysis.

1. **Utility Menu> Select> Entities**

2. (First drop down) "**Elements**"

3. (Second drop down) "**By Attributes**"

4. (Check) "**Elem type num**"

5. "Min,Max,Inc" = 1

6. (Check) "**Unselect**"

7. [Apply].
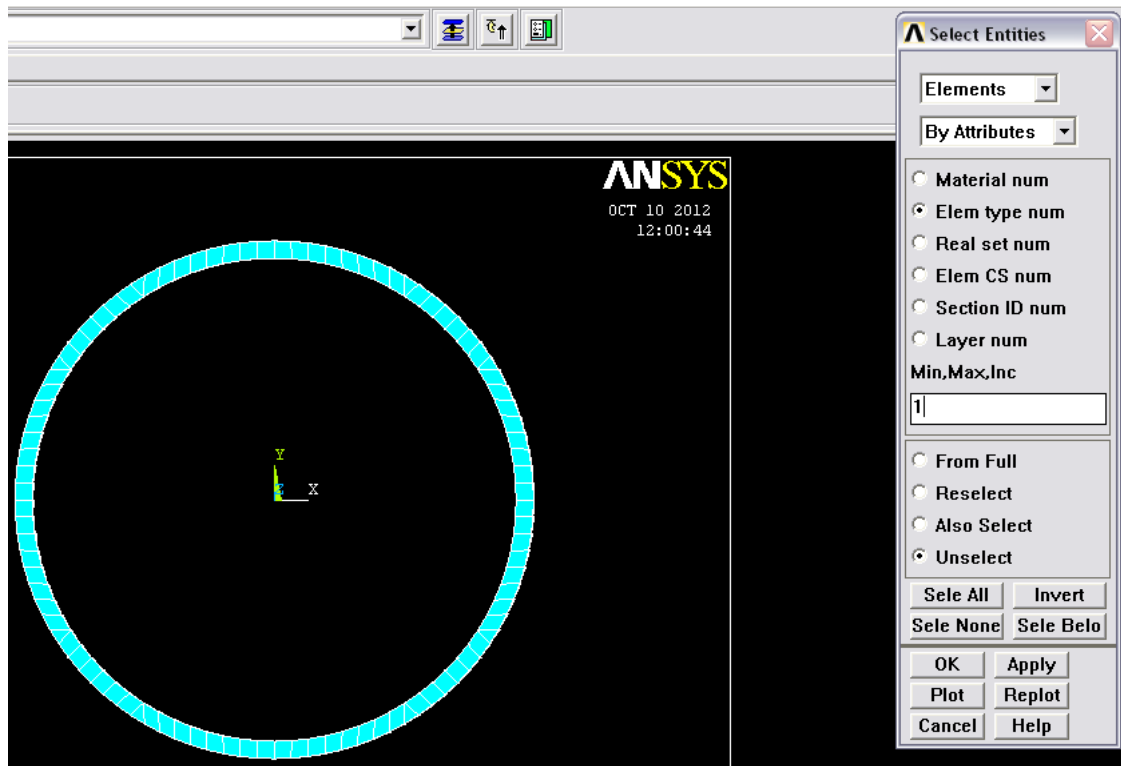
Step 2: Apply constraints to the model.

Constraints will be applied to all nodes according to the end condition. Select all nodes at z =0, then apply the displacement constraints.

1. (First drop down) "**Nodes**"

2. (Second drop down) "**By Location**"

3. (Check) "**Z coordinates**"

4. "Min,Max" = 0

5. (Check) "**From Full**"

6. [Apply]
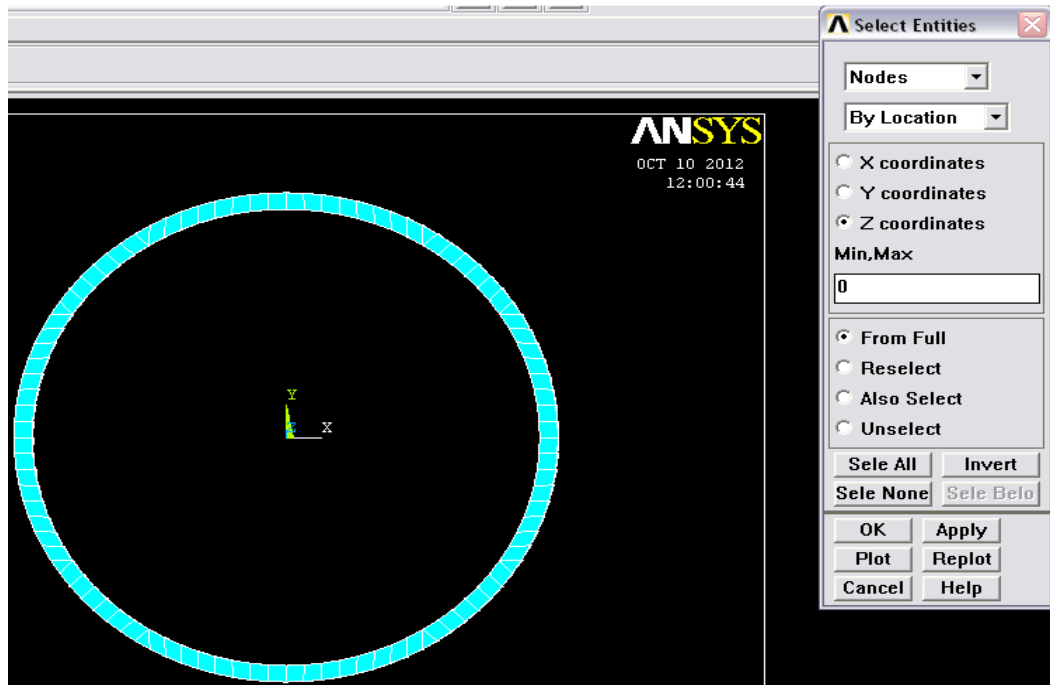
For the constraint of other side:

1. (First drop down) "**Nodes**"

2. (Second drop down) "**By Location**"

3. (Check) "**Z coordinates**"

4. "Min,Max" = 2.2

5. (Check) "**Also Select**"

6. [Apply].
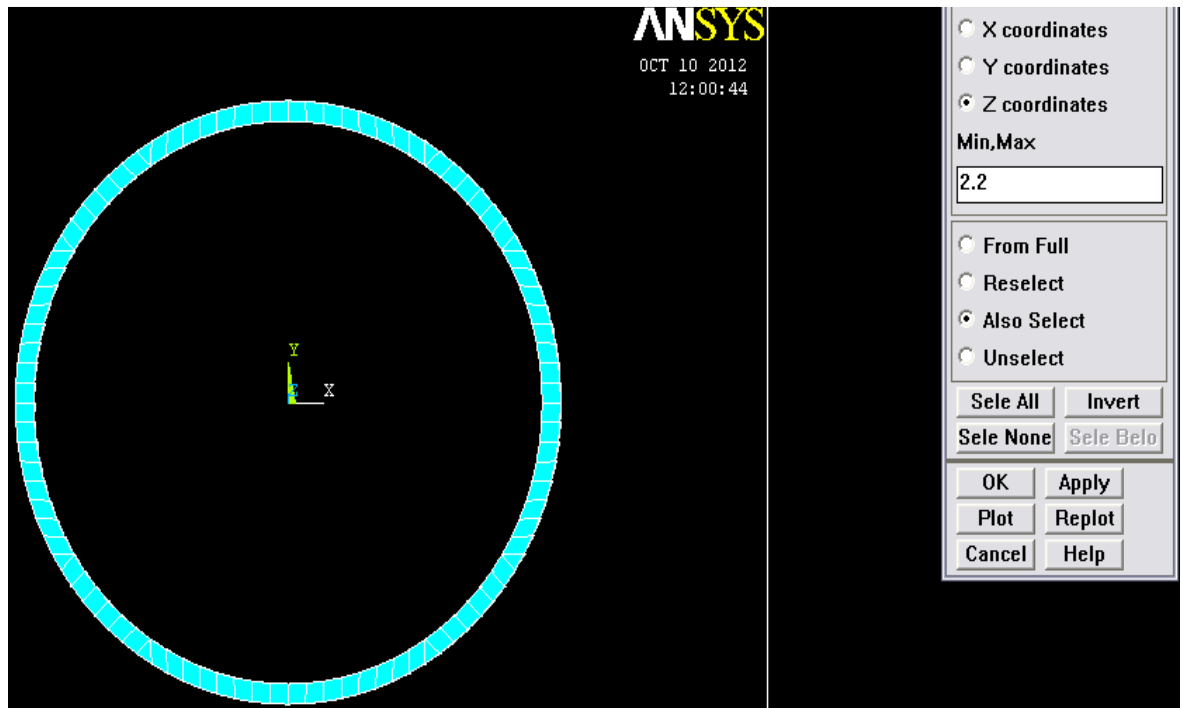
Now the degrees of freedoms are specified at the two ends.

1. **Main Menu> Preprocessor> Loads> Define Loads> Apply> Structural> Displacement> On Nodes**

2. [Pick All] to pick all selected nodes.

3. "DOFs to be constrained" = **DX** and **DY**

4. [OK] Note that by leaving "**Displacement**" blank, a default value of zero is used.

Now, reselect all nodes.

5. (Second drop down) "**By Num/Pick**"

6. [**Sele All**] to immediately select all nodes from entire database.

7. [Cancel] to close dialog box.

Now, before obtaining the solution the nodes of the whole structure are selected again.
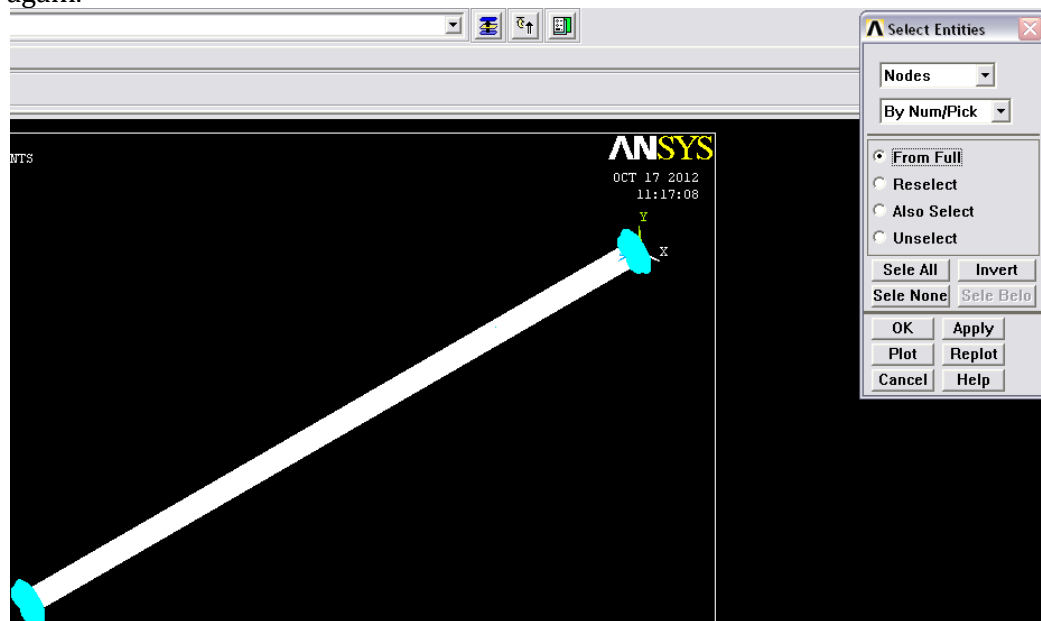
Step 3: Obtaining the Solution:

Specifying analysis type and options

We specified that the type of analysis is modal.

1. **Main Menu> Solution> Analysis Type> New Analysis**

2. (Check) "**Modal**"

3. [OK]
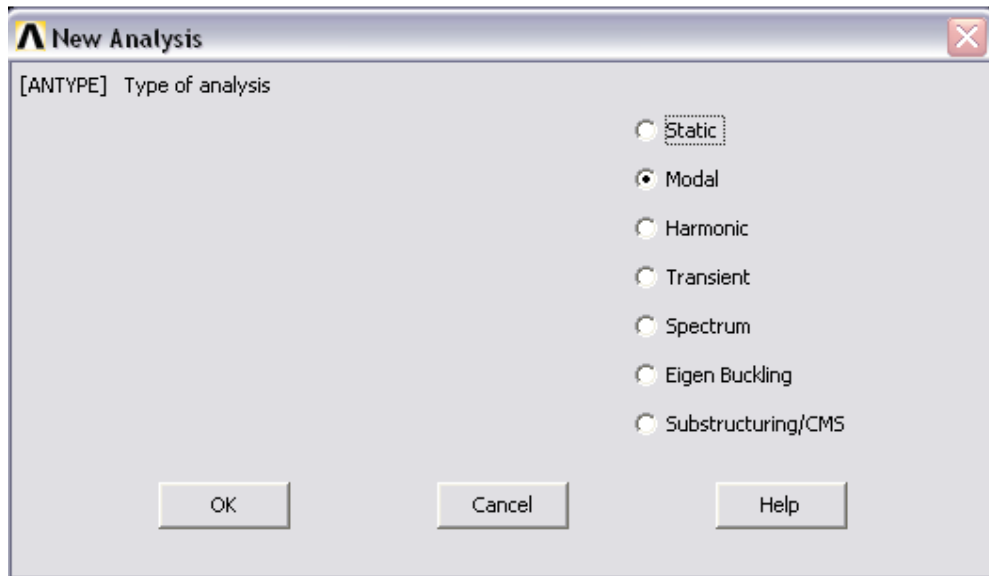
4. **Main Menu> Solution> Analysis Type> Analysis Options**

5. (Check) "**Block Lanczos**" (Block Lanczos is the default for a modal analysis.)

6. "No. of modes to extract" = 20

7. "No. of modes to expand" = 20

8. [OK]

9. [OK] all default values are acceptable for this analysis.
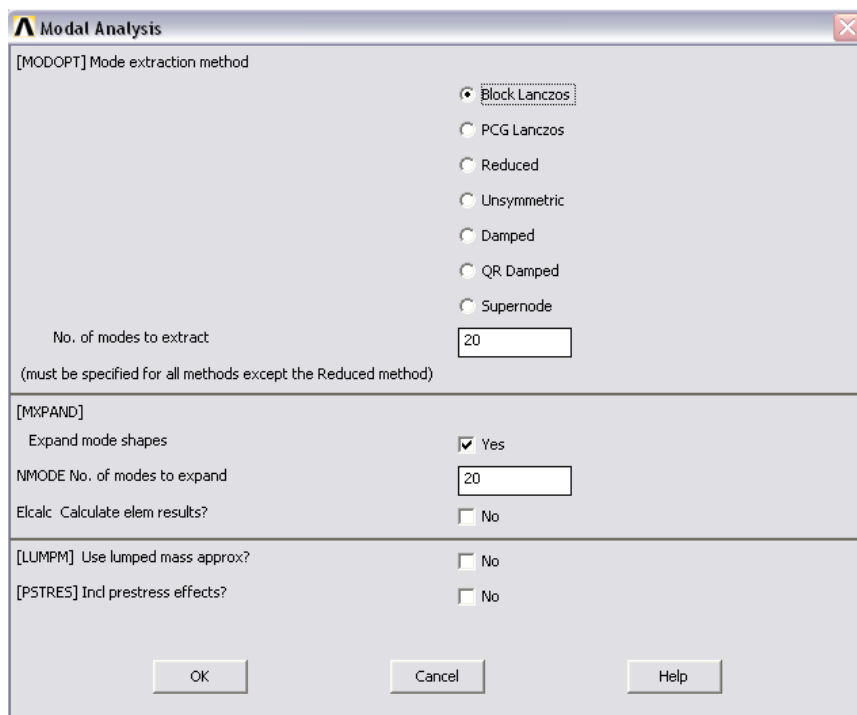
Step 4: Solve.

1. **Main Menu> Solution> Solve> Current LS**

2. Review the information in the status window, and then choose: **File> Close** (Windows).
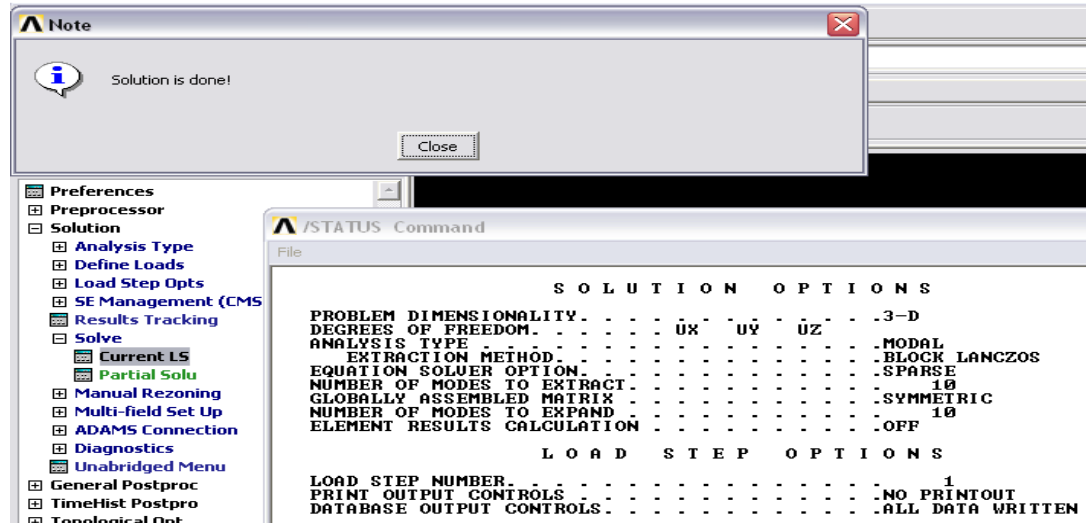
REVIEWING THE RESULTS

Listing the natural frequencies:

1. **Main Menu> General Postproc> Results Summary.**

2. [Close] after observing the listing.
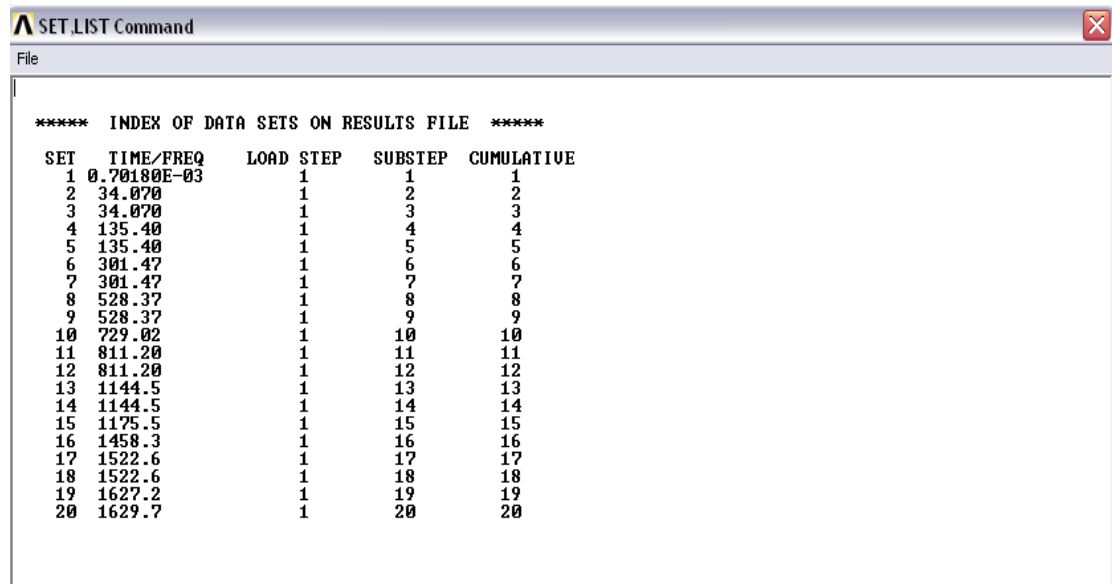
Animating the mode shapes:

To animate the mode shapes at first the corresponding set should be read.
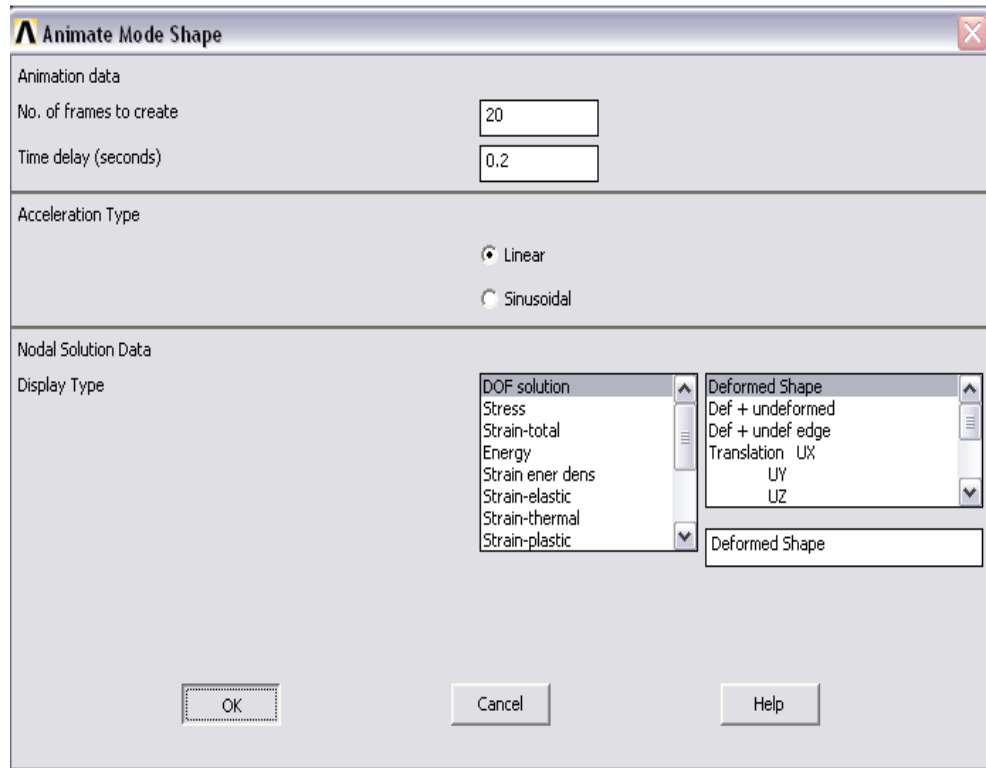
FIRST MODE:

1. Main Menu> General Postproc> Read Results> First Set

2. Utility Menu> PlotCtrls> Animate> Mode Shape.

SECOND MODE:

1. Main Menu> General Postproc> Read Results> Next Set

2. Utility Menu> PlotCtrls> Animate> Mode Shape.
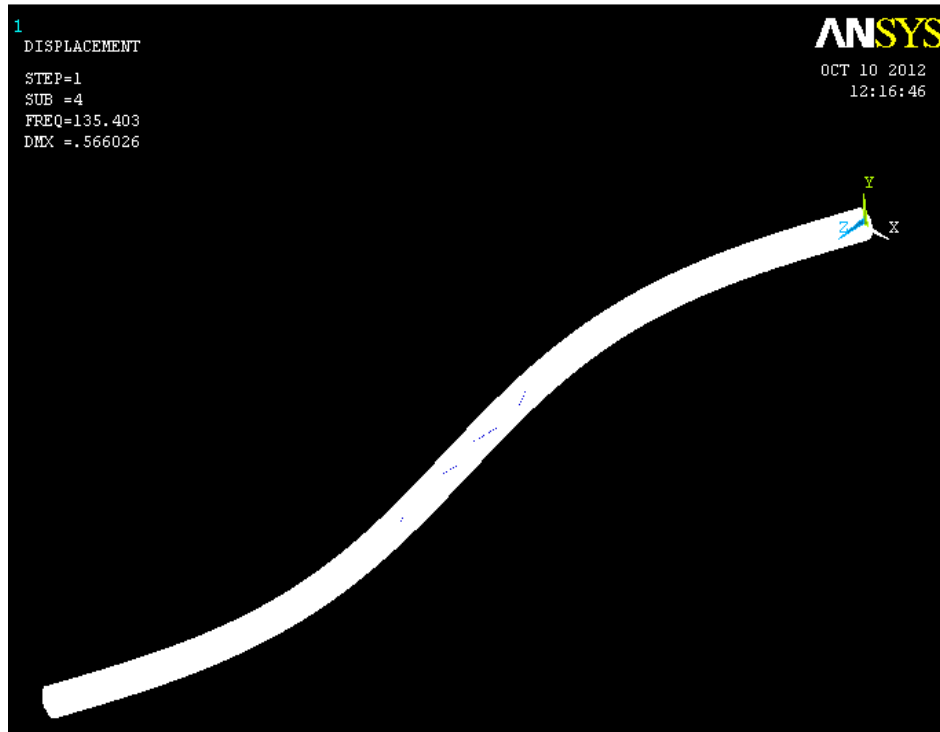
THIRD MODE:

1. Main Menu> General Postproc> Read Results> Next Set

2. Utility Menu> PlotCtrls> Animate> Mode Shape.
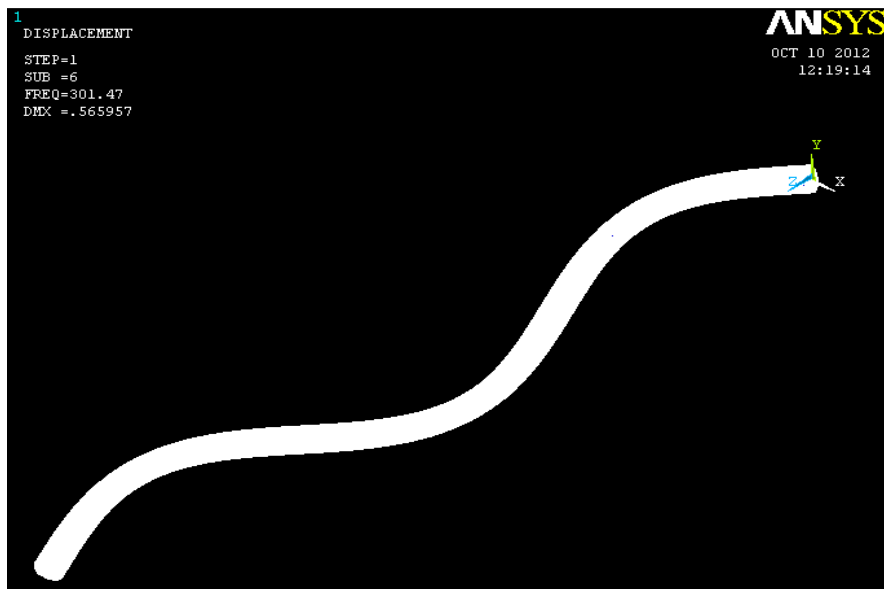
FOURTH MODE:

1. Main Menu> General Postproc> Read Results> Next Set

2. Utility Menu> PlotCtrls> Animate> Mode Shape.
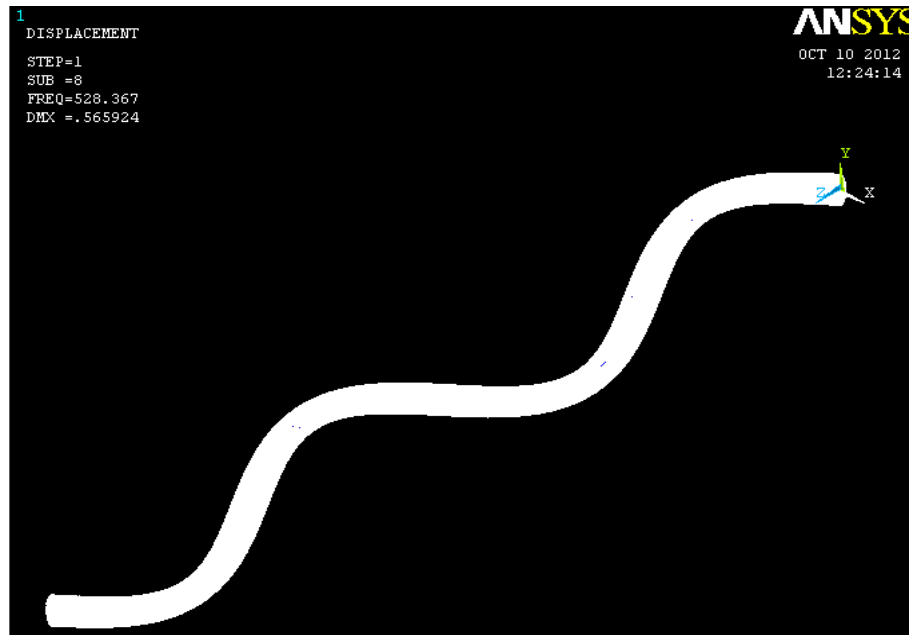
FIFTH MODE:

1. Main Menu> General Postproc> Read Results> Next Set

2. Utility Menu> PlotCtrls> Animate> Mode Shape.
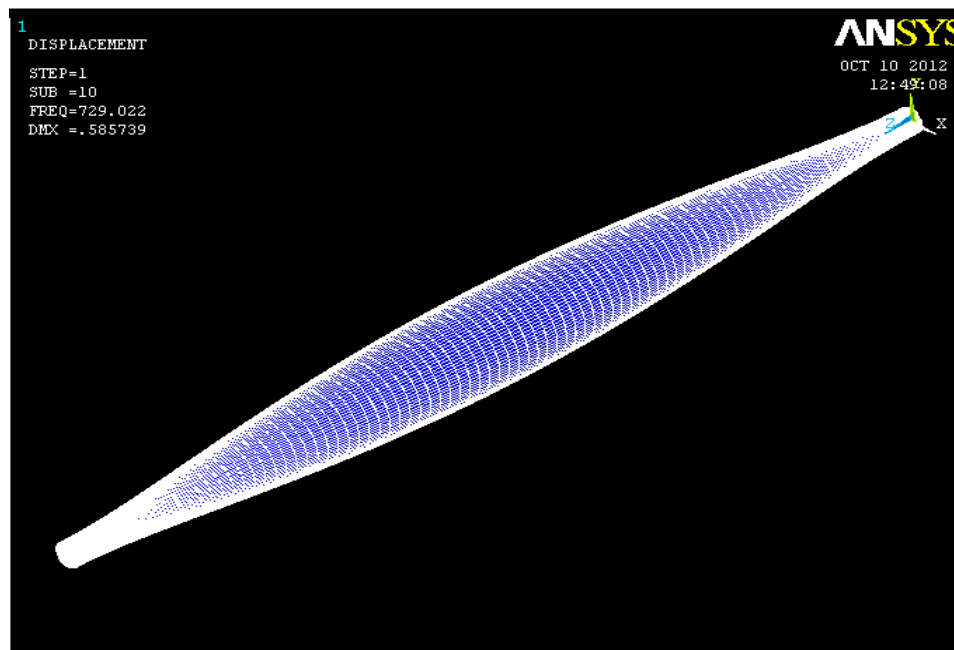
SIXTH MODE:

1. Main Menu> General Postproc> Read Results> Next Set

2. Utility Menu> PlotCtrls> Animate> Mode Shape.
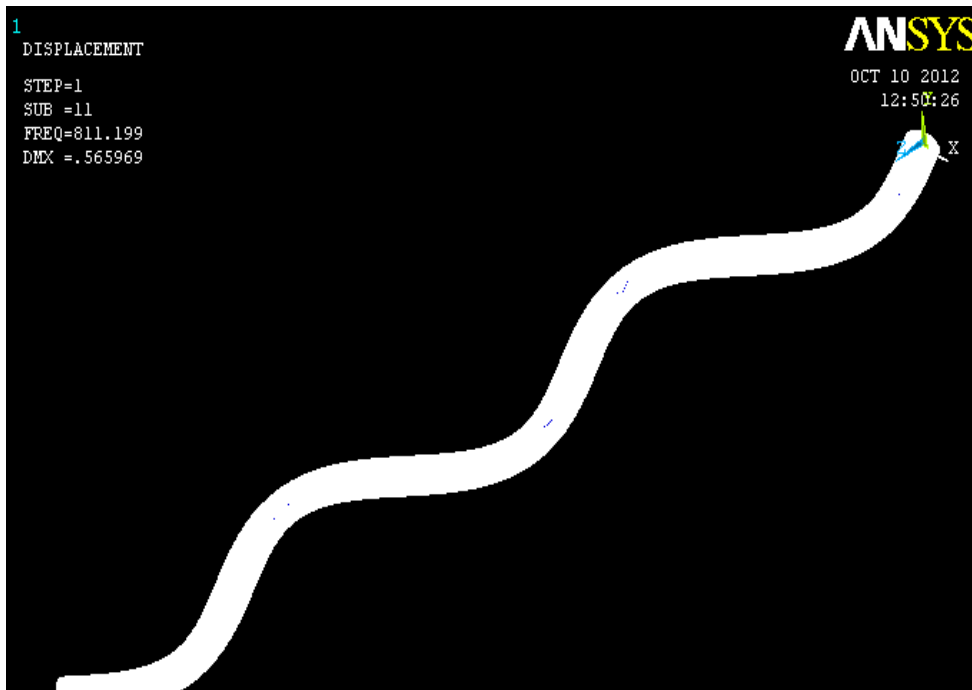
USE OF ANSYS IN HARMONIC ANALYSIS

Before performing the harmonic analysis the model is drawn again as previously done in modal analysis and is meshed.

1. Define Analysis Type (Harmonic)

Solution >Analysis Type >New Analysis >Harmonic.
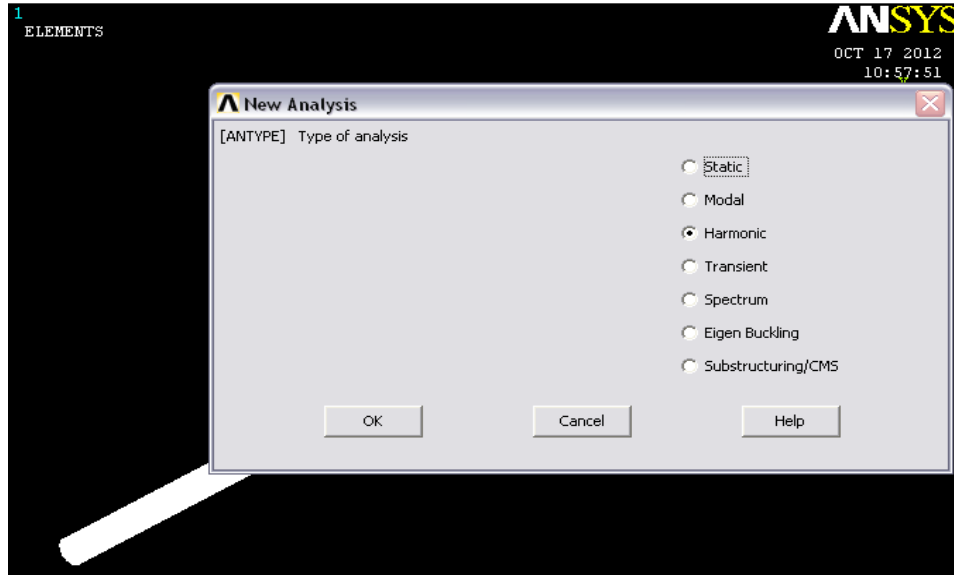
2. Setting options for analysis type:

Select: **Solution > Analysis Type > Analysis Options**. The following window will appear
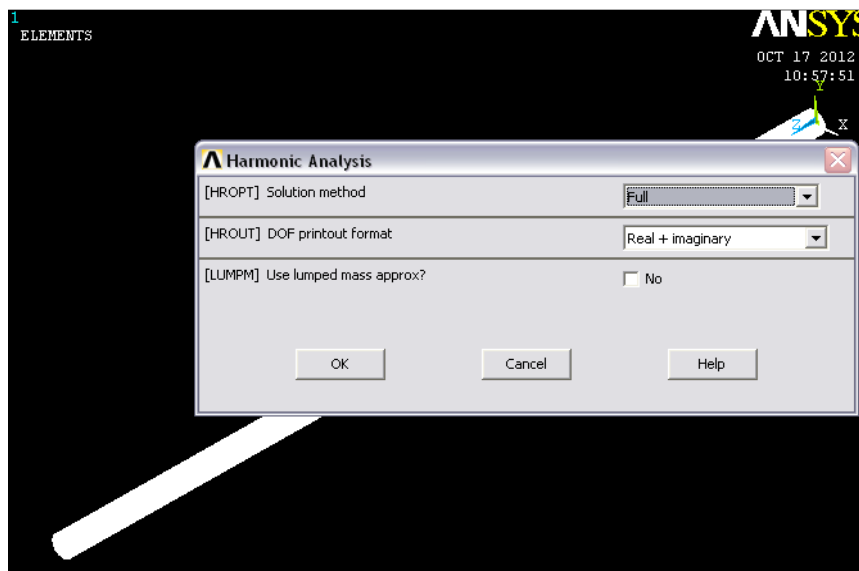
Clicking 'OK' will make the following window appear. The default settings (shown below) are used.
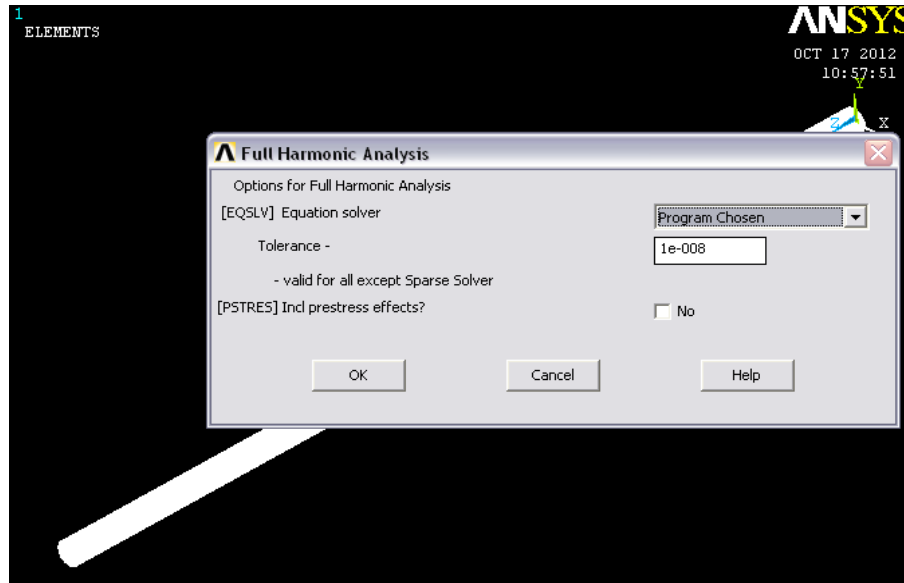
3. Force application

At first the middle node is selected. **Main menu > select> entities**. Then
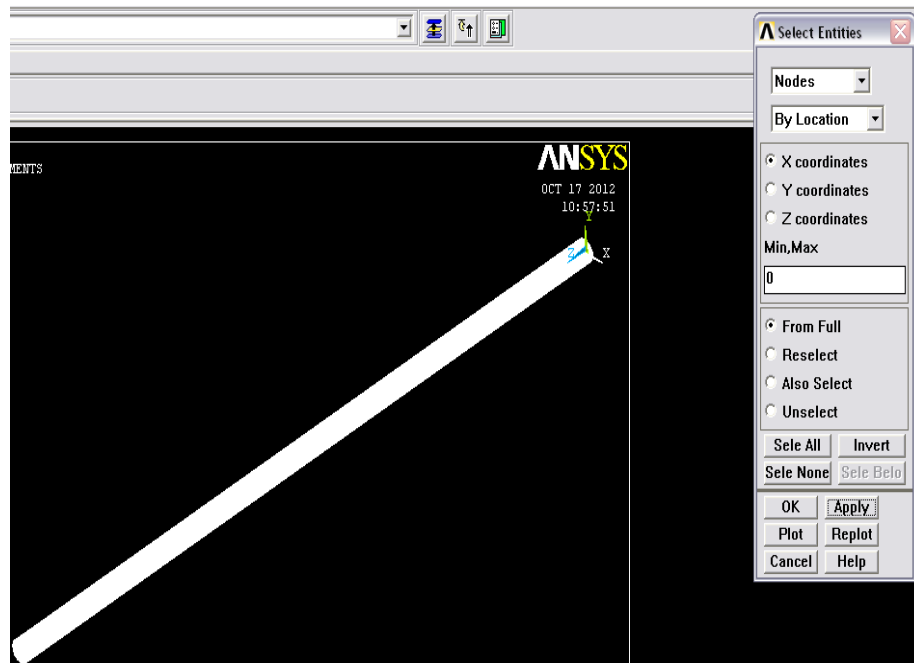
Then,
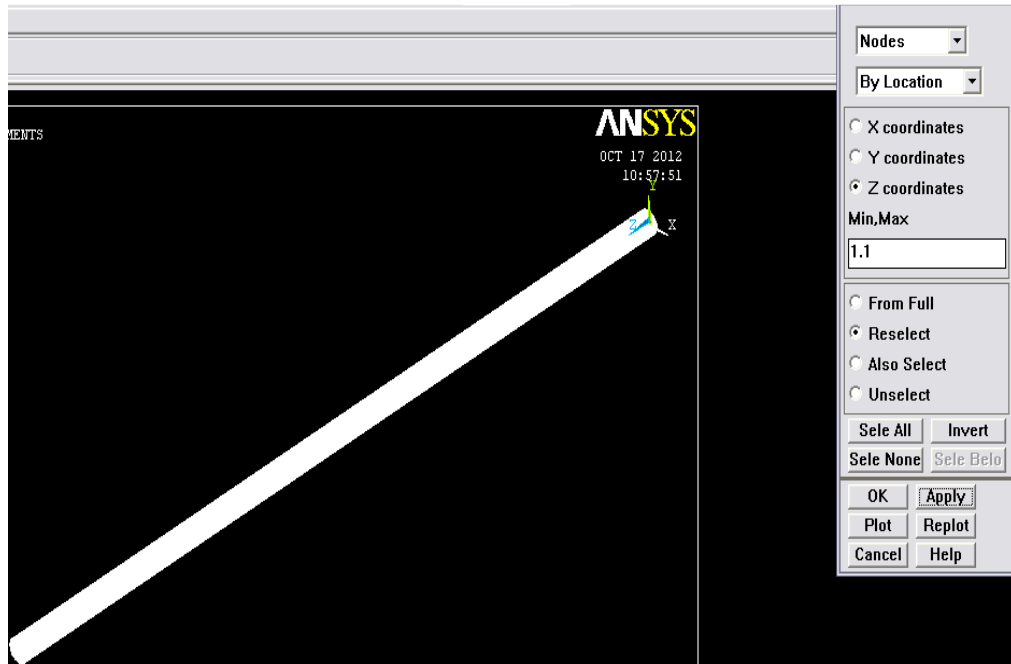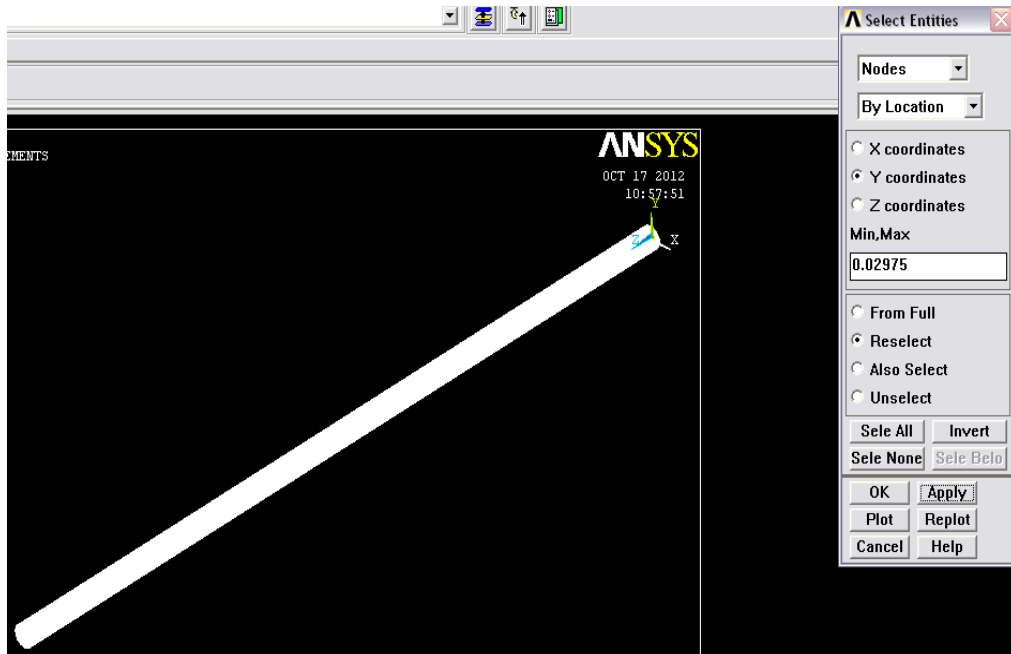
The selected node can be visualized in the screen by: **Main menu> Plot> Nodes**. It will show the selected node. Then the force is applied as such: **Solution > Define Loads > Apply > Structural > Force/Moment > on Nodes**. Selecting **'pick all'** selected the only visible node.
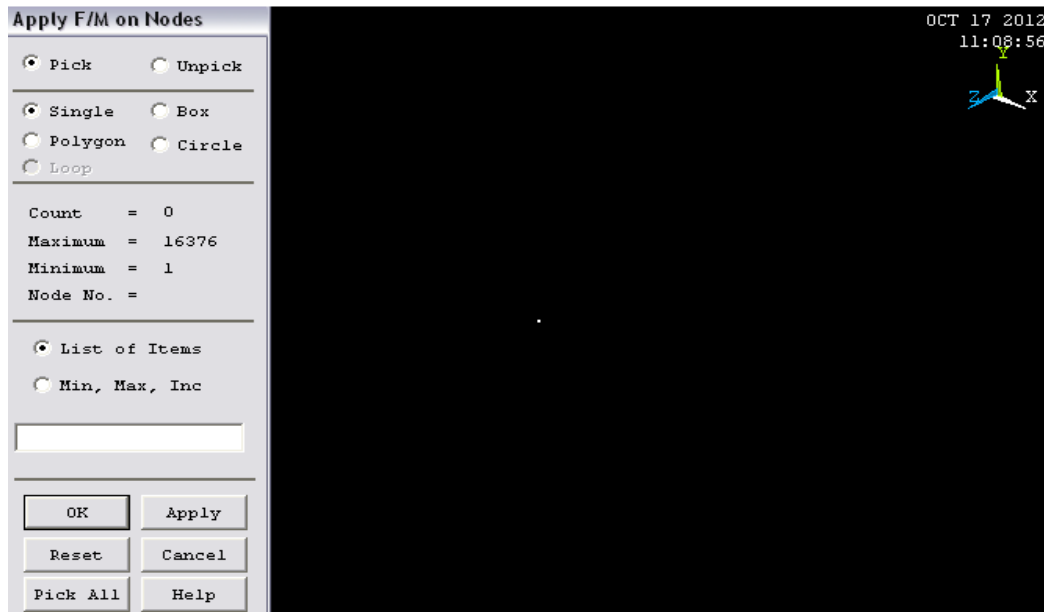
Then the amount of force applied is specified. Here direction of force applied was along **Y** direction and amount of force applied was **10** N.
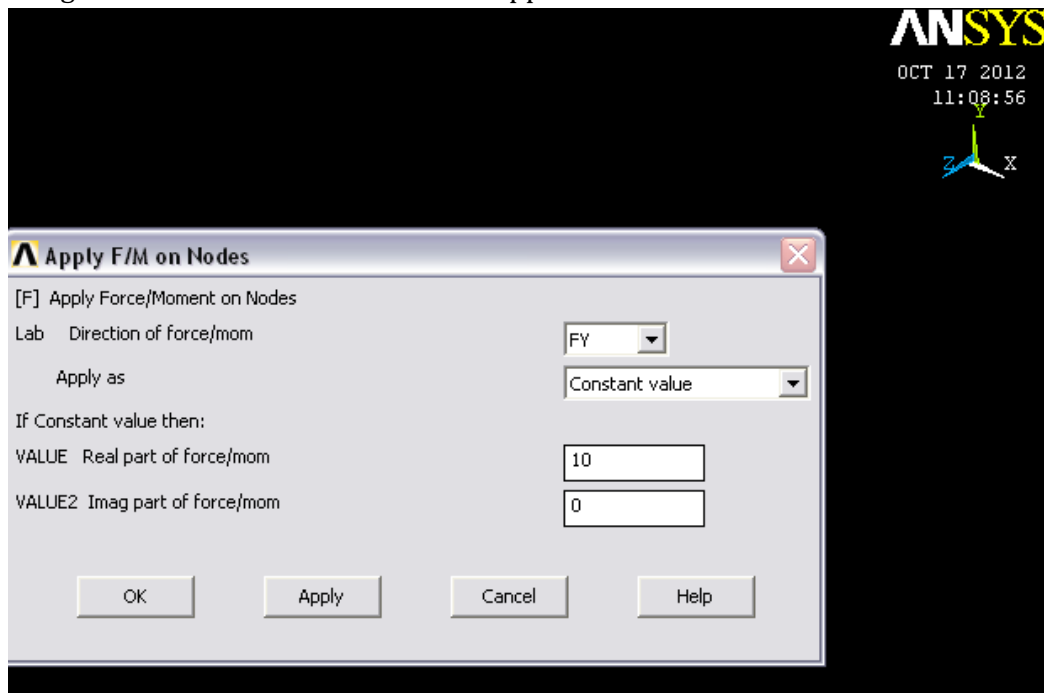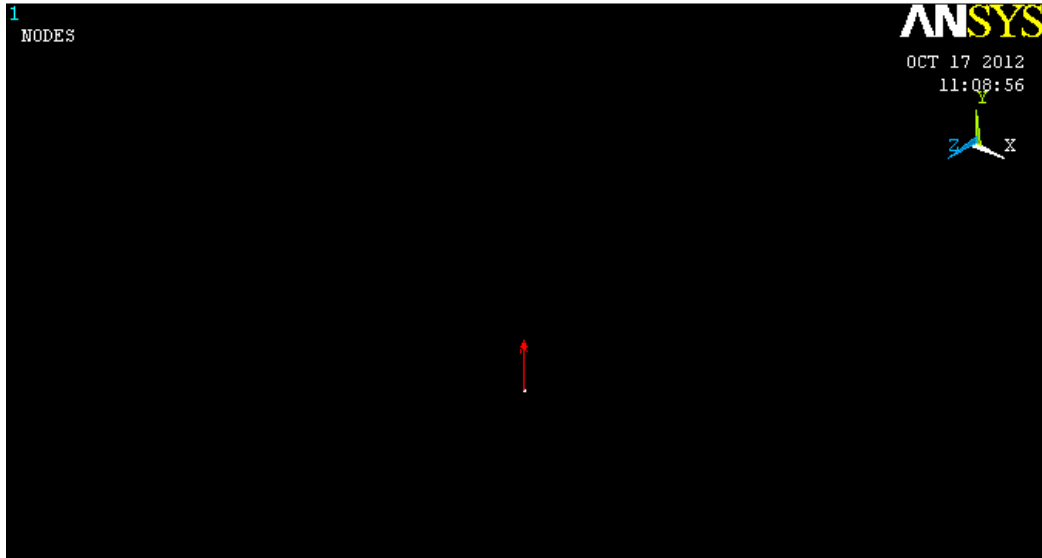
The above snapshot actually shows the application of force in the direction of y axis. Now all the nodes should be selected again to apply constrains at the two furthest ends of the shell.
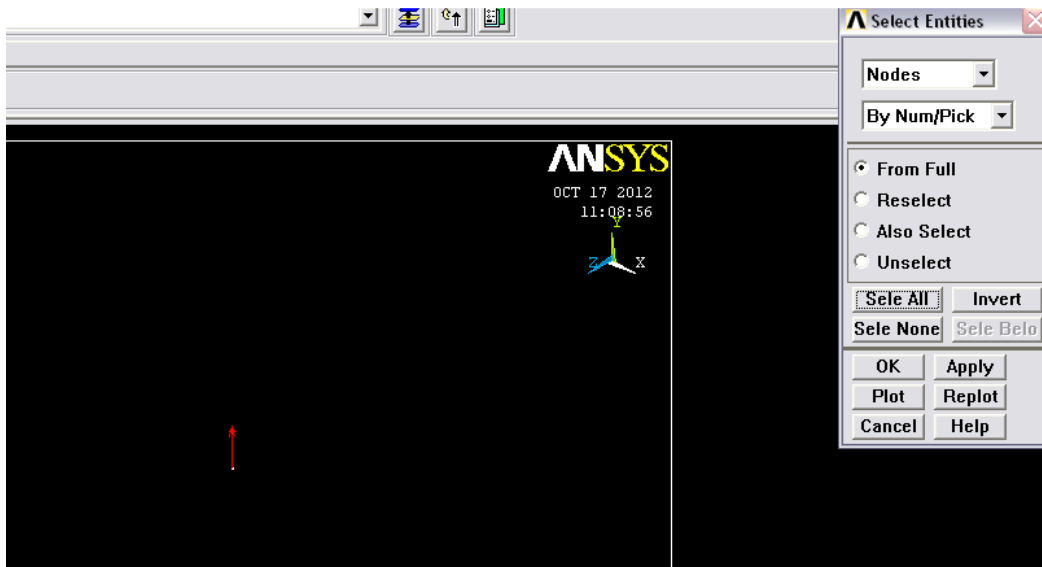
After all the nodes are selected then constraints are applied. Element can be selected as: main **menu > plot> elements**.

After the elements were selected to view complete shell structure is shown in the screen with a red arrow on the top of the shell just at the middle showing the direction of the application of force.
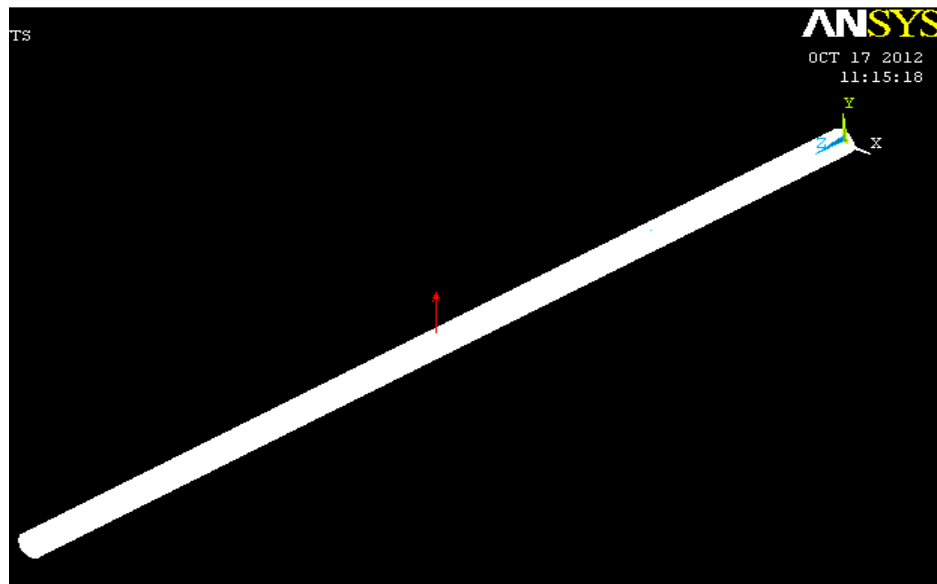
4. Applying Constraints:

To apply the constraints the nodes where the constraints are to be applied should be selected.

Nodes of one end: **Main menu>Select entities> (First drop down) Nodes> (second dropdown) By location> (Select) Z coordinates.**

Selecting the **From Full**, 0 is placed after **Max, Min**.
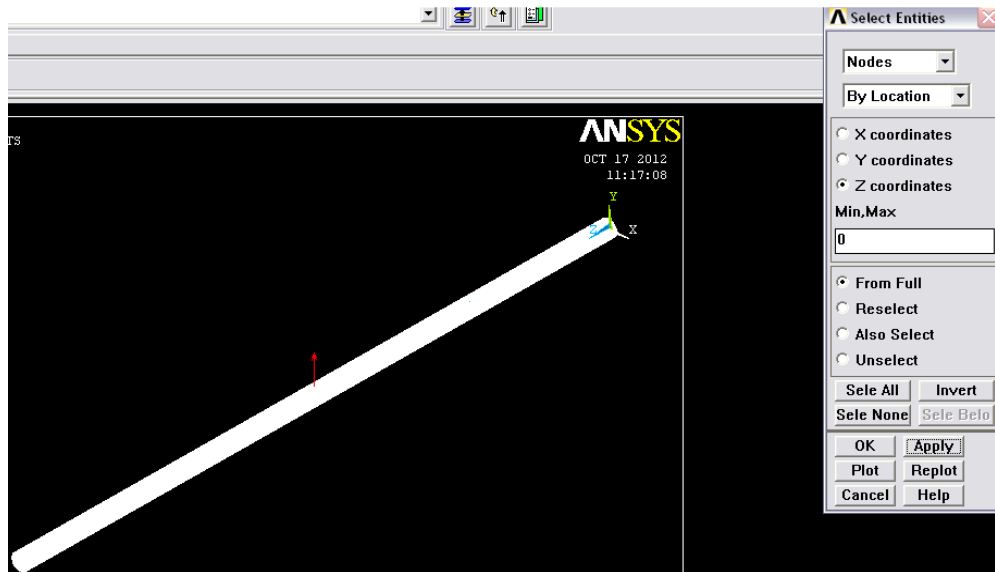
That is nodes at one end are already being selected. Now in almost similar way nodes at other end i.e. at length **2.2** m are to be selected.

Nodes of other end: **Main menu>Select entities> (First drop down) Nodes> (second dropdown) By location> (Select) Z coordinates.**

Selecting the '**Also Select**', **2.2** is placed after **Max, Min**.

Thus nodes at both ends are selected. Now constraint is applied: **Solution > Define Loads > Apply > Structural > Displacement > On Nodes.**
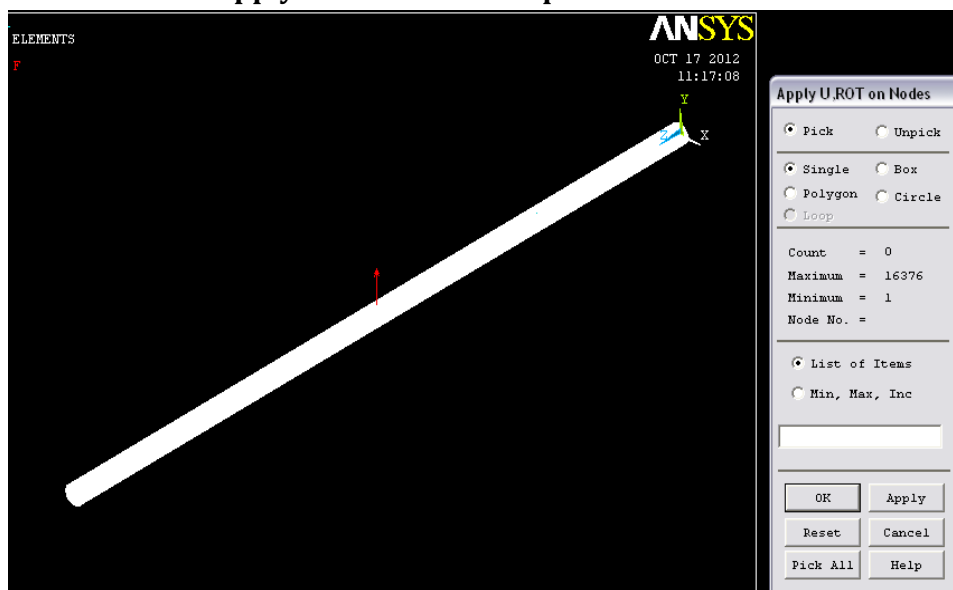
X and Y degrees of freedoms are being constrainted.
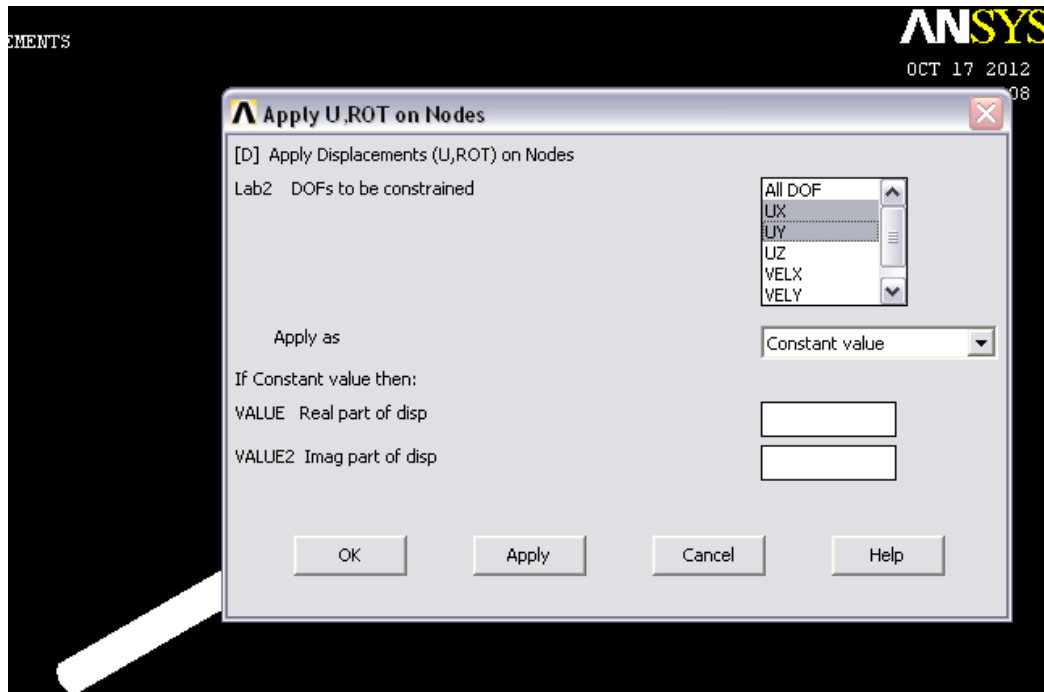
And the setup looks like below-
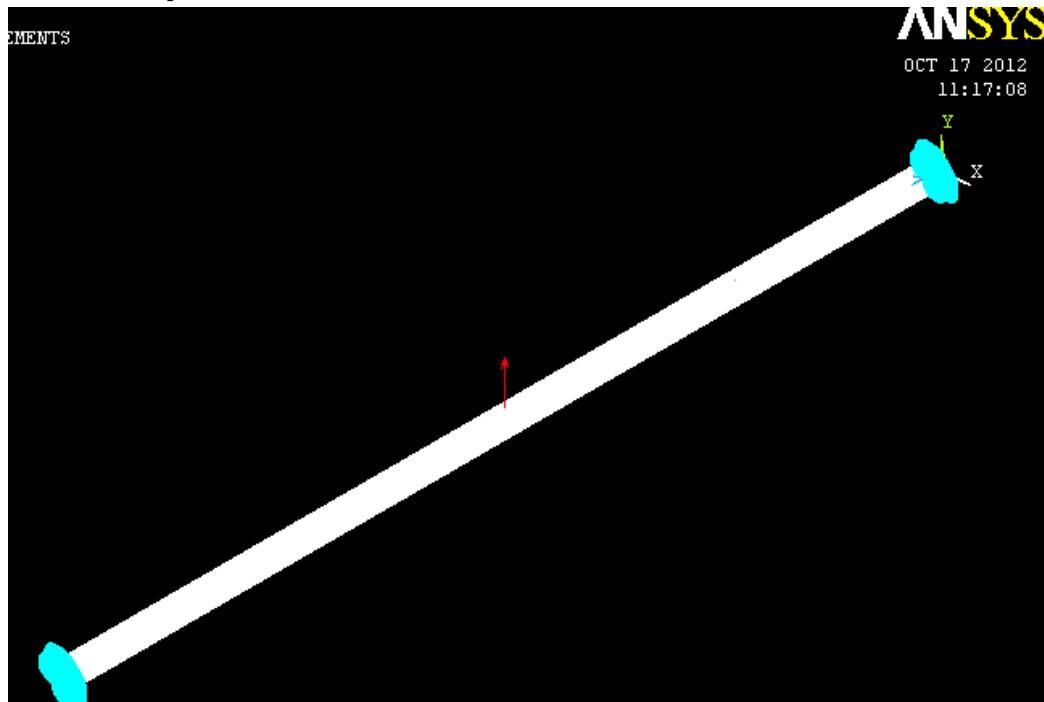
Now for the analysis the nodes of the whole structure are to be selected. **Main menu>Select entities> (First drop down) Nodes > (second dropdown) By Num/Pick.** Then '**From Full'** is selected and '**Sele all'** is pressed with mouse button.
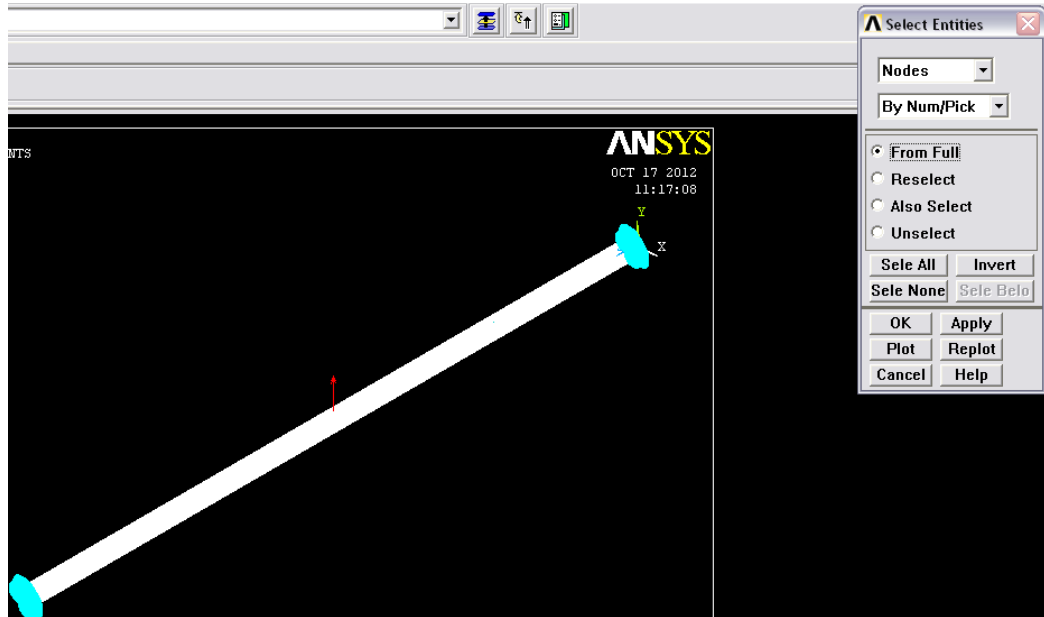
5. Obtaining the solution:

Frequency steps and sub steps options are specified at first-

Frequency range: 0-50 Hz.

Number of substeps: to specify steps within the frequency range
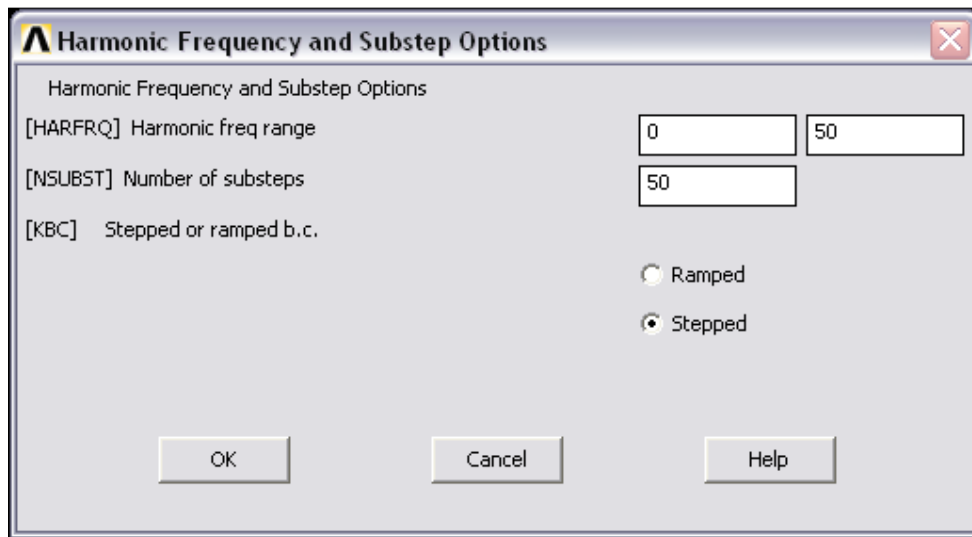
Solving: **Solution > Solve > Current LS.**
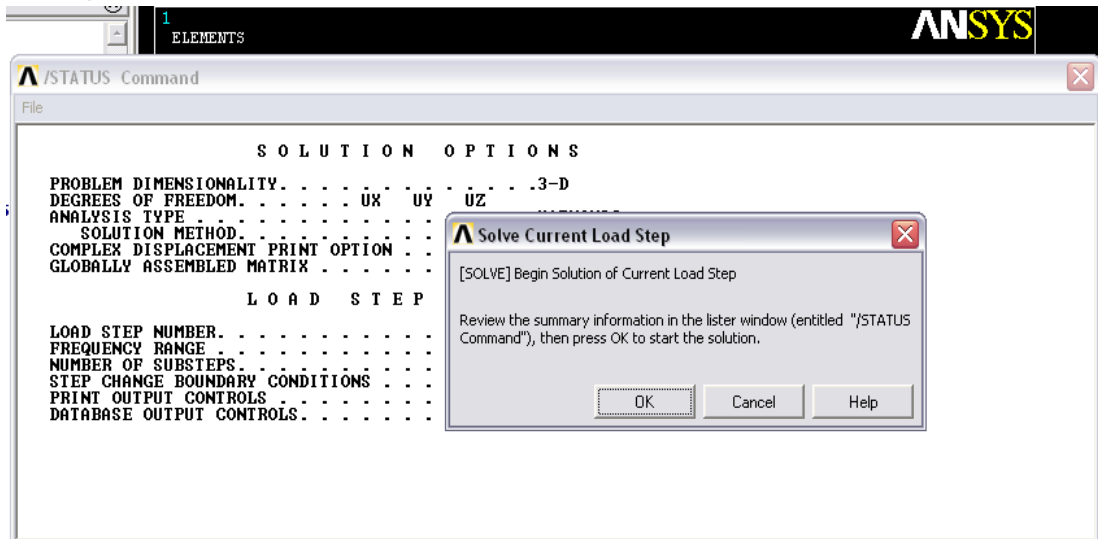
Verification: **Yes.**
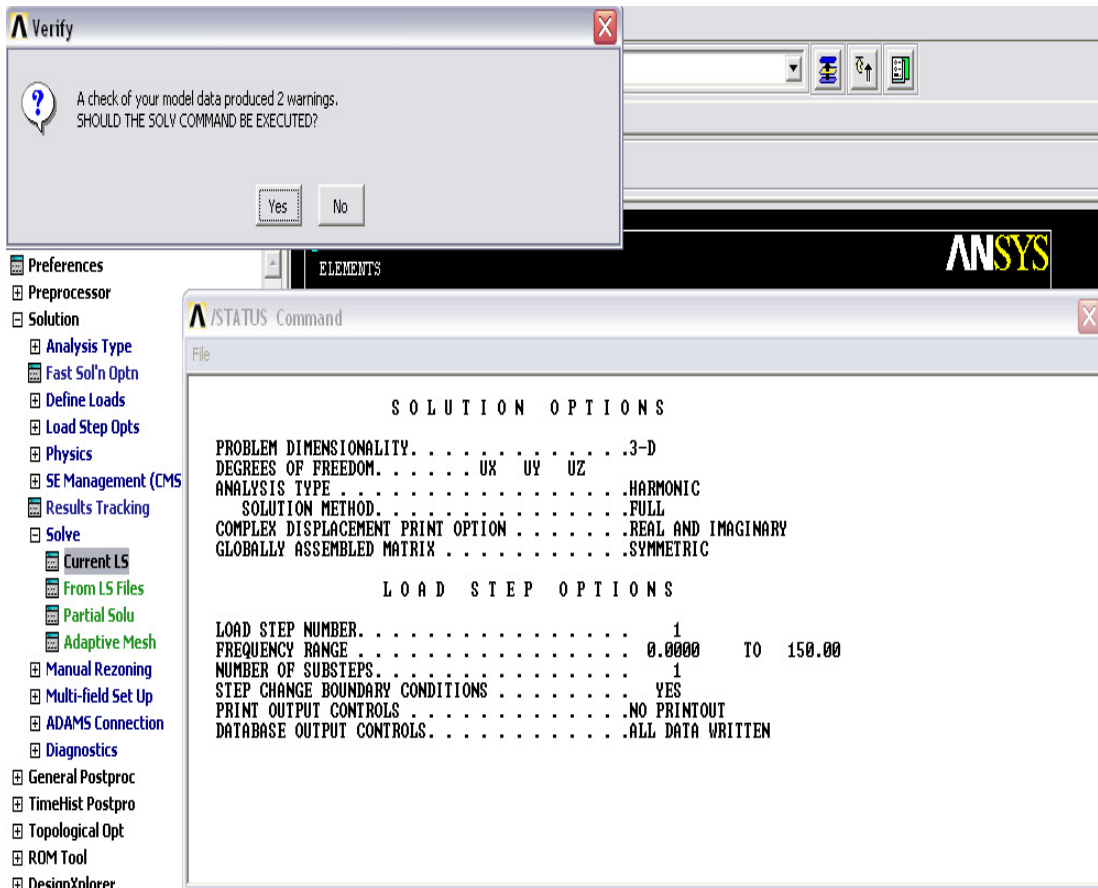
When the solution is done **TimeHist PostPro** is used to see the response of any specific point on the structure.

After selecting the point of force application as the point of investigation through the **Variable Viewer** the graph is plotted taking amplitude as y- axis and frequency as x- axis.
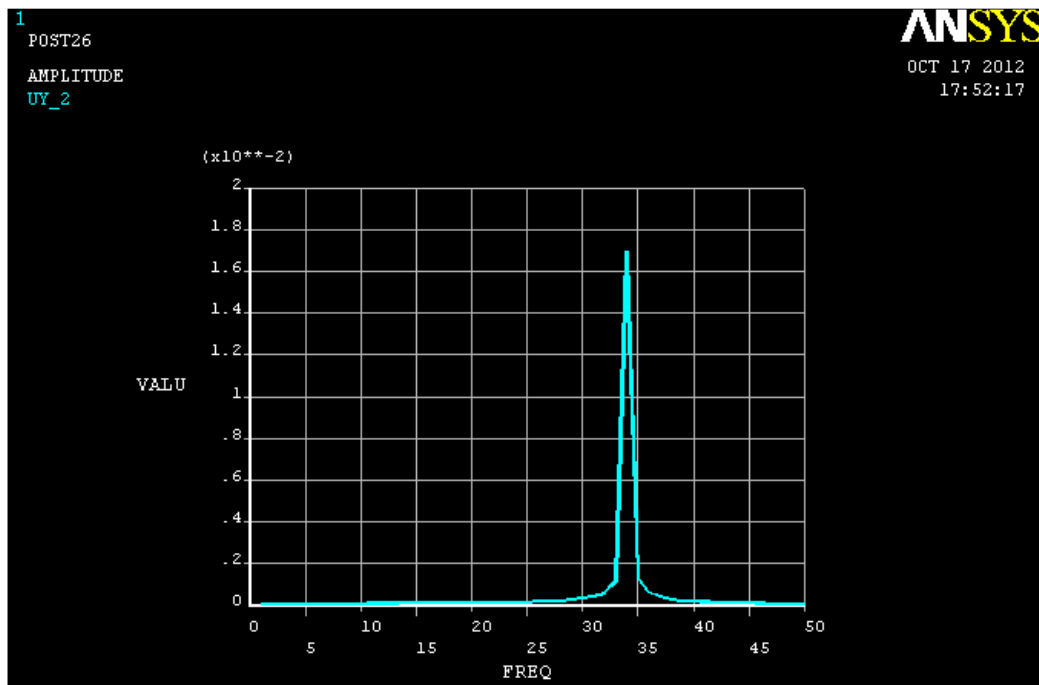
This is amplitude Vs frequency (of applied force) curve.

The list of results-

Frequency   Amplitude   Phase

  1.0000   0.716047E-04 0.00000

  2.0000   0.717860E-04 0.00000

  3.0000   0.720904E-04 0.00000

  4.0000   0.725209E-04 0.00000

  5.0000   0.730823E-04 0.00000

  6.0000   0.737806E-04 0.00000

  7.0000   0.746237E-04 0.00000

  8.0000   0.756213E-04 0.00000

9.0000   0.767854E-04 0.00000

10.000   0.781307E-04 0.00000

11.000   0.796747E-04 0.00000

12.000   0.814390E-04 0.00000

13.000   0.834495E-04 0.00000

14.000   0.857378E-04 0.00000

15.000   0.883427E-04 0.00000

16.000   0.913119E-04 0.00000

17.000   0.947047E-04 0.00000

18.000   0.985955E-04 0.00000

19.000   0.103079E-03 0.00000

20.000   0.108277E-03 0.00000

21.000   0.114348E-03 0.00000

22.000   0.121507E-03 0.00000

23.000   0.130041E-03 0.00000

24.000   0.140357E-03 0.00000

25.000   0.153039E-03 0.00000

26.000   0.168962E-03 0.00000

27.000   0.189494E-03 0.00000

28.000   0.216909E-03 0.00000

29.000   0.255271E-03 0.00000

30.000   0.312643E-03 0.00000

31.000   0.407591E-03 0.00000

32.000   0.594557E-03 0.00000

33.000   0.113149E-02 0.00000

34.000   0.170011E-01 0.00000

35.000   0.126071E-02 180.000

36.000   0.597904E-03 180.000

37.000   0.387697E-03 180.000

38.000   0.284582E-03 180.000

39.000   0.223386E-03 180.000

40.000   0.182901E-03 180.000

41.000   0.154156E-03 180.000

42.000   0.132710E-03 180.000

43.000   0.116108E-03 180.000

44.000   0.102885E-03 180.000

45.000   0.921125E-04 180.000

46.000   0.831735E-04 180.000

47.000   0.756414E-04 180.000

48.000   0.692125E-04 180.000

49.000   0.636642E-04 180.000

50.000   0.588301E-04 180.000

It clearly shows that when frequency of forced vibration reaches 1st natural frequency the first resonance takes place i.e. amplitude increases dramatically and later the response goes out of phase with the force and most of the structures cannot sustain this high amplitude vibration.

CHAPTER 4

ANALYTICAL ANALYSIS FOR DETERMINING NATURAL FREQUENCIES AND FORCED VIBRATION RESPONSE OF CIRCULAR CYLINDRICAL SHELL

In this chapter, we will look into a mathematical approach, as described by **Werner Soedel** in the book titled **"Vibrations of Shells and Plates, Third Edition"** to find out the natural frequencies of a simply supported circular cylindrical shell and see that how the structure responds being subjected to a harmonically varying load or a sinusoidal force. For that, equations of motion for the shell are needed which requires a way to relate the motion of the structure to the loads acting on it. This is done by first developing relationships between stress and strain, as strain is technically deformation of the body so stresses can be related to displacement of the structure which again warrants for the relationships between strain and displacement. Augustus Edward Hough Love, using Hamilton's principle, defined motion under any type of pressure load. Love's equations can be used to determine equations of motion for different shell type structures. Love extended work on shell vibration by Rayleigh, by considering the coexistence of both the two classes of shells, defined by Rayleigh earlier. These are –

1. One where the middle surface does not stretch and bending effects are the only important ones, and

2. One where only the stretching of the middle surface is important and the bending stiffness can be neglected.

The basic assumptions made for the following sections,

1. Shells are thin with respect to their radii of curvature, and
2. Deflections on the shell are reasonably small.

SHELL COORDINATES AND INFINITESIMAL DISTANCES IN SHELL LAYERS

Here we will assume that thin, isotropic, and homogeneous shells of constant thickness have neutral surfaces, just as beams in transverse deflection have neutral fibers. Stresses in such a neutral surface can be of the membrane type but cannot be bending stresses. To locate any point on the neural surface of the shell we will use curvilinear coordinate system. The location of point P on the neutral surface in three dimensional Cartesian coordinates can be expressed by two dimensional curvilinear surface coordinates as follows,

$$x_1 = f_1(\alpha_1, \alpha_2), x_2 = f_2(\alpha_1, \alpha_2), \ x_3 = f_3(\alpha_1, \alpha_2) \tag{1}$$

The location of P on the neutral surface can also be expressed by a vector,

$$\bar{r}(\alpha_1, \alpha_2) = f_1(\alpha_1, \alpha_2)\overline{e_1} + f_2(\alpha_1, \alpha_2)\overline{e_2} + f_3(\alpha_1, \alpha_2)\overline{e_3} \tag{2}$$
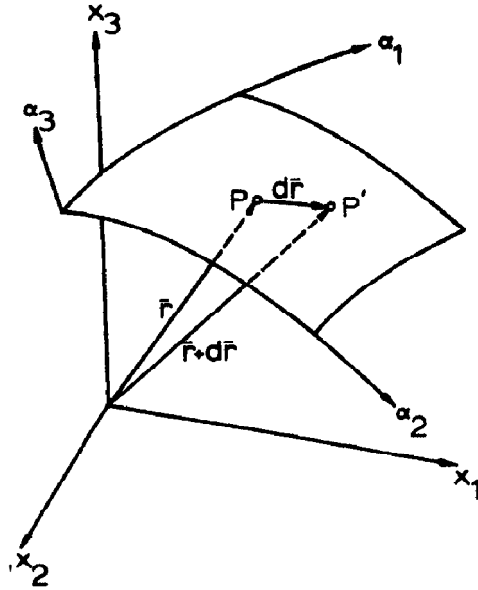
Fig 1. Reference surface

Now the infinitesimal distance between points P and P′ on the neutral surface is the differential change, $d\bar{r}$ of the vector $\bar{r}$ from P to P′,

$$d\bar{r} = \frac{\partial \bar{r}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_2 \tag{3}$$

The magnitude ds of $d\bar{r}$ is given by,

$$(ds)^2 = d\bar{r}.d\bar{r} \tag{4}$$

Simplifying this, we get,

$$(ds)^2 = A_1^2(d\alpha_1)^2 + A_2^2(d\alpha_2)^2 \tag{5}$$

Where,

$$A_1^2 = \frac{\partial \bar{r}}{\partial \alpha_1} \cdot \frac{\partial \bar{r}}{\partial \alpha_1} = \left| \frac{\partial \bar{r}}{\partial \alpha_1} \right|^2 \tag{6}$$

And,

$$A_2^2 = \frac{\partial \bar{r}}{\partial \alpha_2} \cdot \frac{\partial \bar{r}}{\partial \alpha_2} = \left| \frac{\partial \bar{r}}{\partial \alpha_2} \right|^2 \tag{7}$$

This equation is called the "fundamental form" and A_1 & A_2 are the "fundamental form parameters" or "Lamé parameters".

Now being specific to our structure of interest i.e. circular cylindrical shell, for each point on shell surface there are two maximum and minimum radius of curvature, whose directions are perpendicular to each other. These lines of principal curvature are in this case parallel to the axis of revolution, where the radius of curvature $R_x = \infty$ (i.e. curvature $1/R_x = 0$) and along the circles, where the radius of curvature $R_\theta = a$ (i.e. curvature $1/R_\theta = 1/a$).
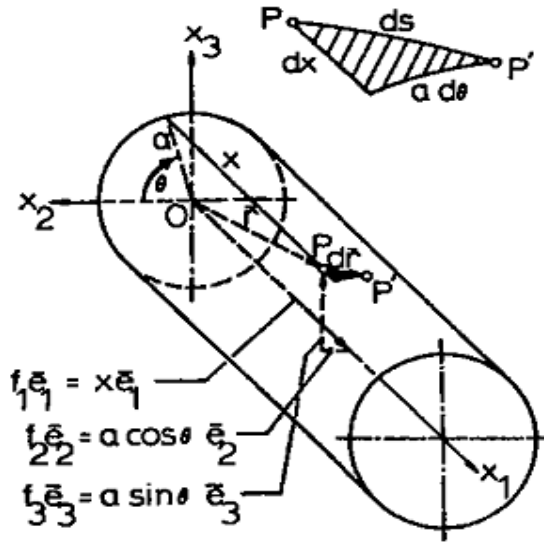
Fig 2. Obtaining Lamé parameters for circular cylindrical shell

Therefore, the curvilinear coordinates are,

$$\alpha_1 = x , \alpha_2 = \theta \tag{8}$$

So, the position vector discussed before (eqn.2) becomes,

$$\bar{r} = x\overline{e_1} + a\cos\theta\overline{e_2} + a\sin\theta\overline{e_3} \tag{9}$$

Thus,

$$\frac{\partial\bar{r}}{\partial\alpha_1} = \frac{\partial\bar{r}}{\partial x} = \overline{e_1} \quad or, \quad \left|\frac{\partial\bar{r}}{\partial\alpha_1}\right| = A_1 = 1 \tag{10}$$

And,

$$\frac{\partial\bar{r}}{\partial\alpha_2} = \frac{\partial\bar{r}}{\partial\theta} = -a\sin\theta\overline{e_2} + a\cos\theta\overline{e_3} \tag{11}$$

$$or, \left|\frac{\partial\bar{r}}{\partial\theta}\right| = A_2 = a\sqrt{\sin^2\theta + \cos^2\theta} = a \tag{12}$$

The fundamental form is therefore,

$$(ds)^2 = (dx)^2 + a^2(d\theta)^2 \tag{13}$$

Recognizing that the fundamental form can be interpreted as defining the hypotenuse ds of a right triangle whose sides are infinitesimal distancesalong the surface coordinates of the shell, we may obtain A_1 and A_2 in a simpler fashion by expressing ds directly using inspection:

$$(ds)^2 = (dx)^2 + a^2(d\theta)^2 \tag{14}$$

By comparison with eqn.3, we get,

$$A_1 = 1, A_2 = a \tag{15}$$

Now let us define α_3 coordinate to be perpendicular to $\alpha_2\alpha_3$ plane i.e. for circular cylindrical shell it is the normal direction to the undeflected shell surface and α_3 is 0 on neutral $\alpha_2\alpha_3$ plane. If P$_1$ and P$_1$' are two points on different $\alpha_2\alpha_3$ planes whose projections on the neutral surface are at infinitesimal distance, then the distance between these two points, ds can be derived by similar mathematical approach to be,

$$(ds)^2 = A_1^2\left(1 + \frac{\alpha_3}{R_1}\right)^2 (d\alpha_1)^2 + A_2^2\left(1 + \frac{\alpha_3}{R_2}\right)^2 (d\alpha_2)^2 + (d\alpha_3)^2 \tag{16}$$

Where, $R_1$and R_2 are radius of curvatures.

This equation gives the distance between two points of an undeflected shell.

STRESS-STRAIN RELATIONSHIP

According to the coordinate system we have chosen, we have three mutually perpendicular planes of strain and three shear strains. We assume that Hooke's law applies, therefore we have for a three dimensional element,

$$\varepsilon_{11} = \frac{1}{E}[\sigma_{11} - \mu(\sigma_{22} + \sigma_{33})] \tag{17}$$

$$\varepsilon_{22} = \frac{1}{E}[\sigma_{22} - \mu(\sigma_{11} + \sigma_{33})] \tag{18}$$

$$\varepsilon_{33} = \frac{1}{E}[\sigma_{33} - \mu(\sigma_{11} + \sigma_{22})] \tag{19}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{G} \tag{20}$$

$$\varepsilon_{13} = \frac{\sigma_{13}}{G} \tag{21}$$

$$\varepsilon_{23} = \frac{\sigma_{23}}{G} \tag{22}$$

Where, E= modulus of elasticity, G= modulus of rigidity, μ= poision's ratio, $\sigma_{11}, \sigma_{22},$ and σ_{33} are normal stresses and $\sigma_{12}, \sigma_{13},$ and σ_{23} are shear stresses. And,

$$\sigma_{12} = \sigma_{21}, \sigma_{13} = \sigma_{31}, \sigma_{23} = \sigma_{32} \tag{23}$$

We will later assume that transverse shear deflections can be neglected.This implies that,

$$\varepsilon_{13} = 0, \varepsilon_{23} = 0 \tag{24}$$

However, we will not neglect the integrated effect of the transverse shear stresses σ_{13} and σ_{23}.
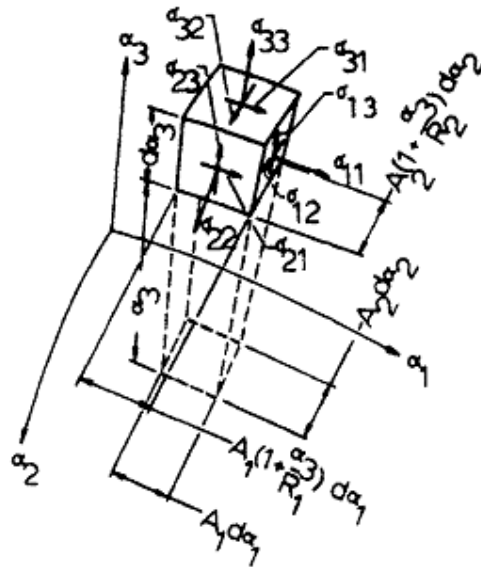
Fig 3. Stresses acting on an element

The normal stress, σ_{33} which acts in the normal direction to the neutral surface, will be neglected,

$$\sigma_{33} = 0 \tag{25}$$

This is because we argue that on an unloaded outer shell surface it is 0, or if a force acts on the shell, it is equivalent in magnitude to the external load on the shell, which is a relatively small value in most cases. Only in the close vicinity of a concentrated load do we reach magnitudes that would make the consideration of σ_{33} worthwhile. Our equation system therefore reduces to,

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \mu\sigma_{22}) \tag{26}$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \mu\sigma_{11}) \tag{27}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{G} \tag{28}$$

And,

$$\varepsilon_{33} = -\frac{\mu}{E}(\sigma_{11} + \sigma_{22}) \tag{29}$$

Only the first three relationships will be of importance in the following. Equation (29) can later be used to calculate the constriction of the shell thickness during vibration, which is of some interest to acousticians since it is an additional noise generating mechanism, along with transverse deflection.

STRAIN-DISPLACEMENT RELATIONSHIPS

In a preceding section we have seen that the distance between two points P_1 and P_1' of an undeflected shell,

$$(ds)^2 = \sum_{i=1}^{3} g_{ii}(\alpha_1, \alpha_2, \alpha_3)(d\alpha_i)^2 \tag{30}$$

Where, short notations used are,

$$A_1^2\left(1 + \frac{\alpha_3}{R_1}\right)^2 = g_{11}(\alpha_1, \alpha_2, \alpha_3) \tag{31}$$

$$A_2^2\left(1 + \frac{\alpha_3}{R_2}\right)^2 = g_{22}(\alpha_1, \alpha_2, \alpha_3) \tag{32}$$

$$1 = g_{33}(\alpha_1, \alpha_2, \alpha_3) \tag{33}$$

Now, if point P_1 originally located at$(\alpha_1, \alpha_2, \alpha_3)$, is deflected in the α_1 direction by U_1, in the α_2 direction by U_2, and in the α_3 direction by U_3, it will be located at $(\alpha_1 + \xi_1, \ \alpha_2 + \xi_2, \ \alpha_3 + \xi_3)$. Deflections U_i and coordinate changes ξ_i are related by,

$$U_i = \sqrt{g_{ii}(\alpha_1, \alpha_2, \alpha_3)}\xi_i \tag{34}$$

And, pointP_1', originally at $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, \alpha_3 + d\alpha_3)$, will be located at $(\alpha_1 + d\alpha_1 + \xi_1 + d\xi_1, \alpha_2 + d\alpha_2 + \xi_2 + d\xi_2, \alpha_3 + d\alpha_3 + \xi_3 + d\xi_3)$ after deflection. The distance, ds' between P_1 and P_1' in the deflected state will therefore be,

$$(ds')^2 = \sum_{i=1}^{3} g_{ii}(\alpha_1 + \xi_1, \alpha_2 + \xi_2, \alpha_3 + \xi_3)(d\alpha_i + d\xi_i)^2 \tag{35}$$

$g_{ii}(\alpha_1, \alpha_2, \alpha_3)$ varies continuously as α_1, α_2 and α_3 change, now taking Taylor series expansion of $g_{ii}(\alpha_1 + \xi_1, \alpha_2 + \xi_2, \alpha_3 + \xi_3)$ about the point $(\alpha_1, \alpha_2, \alpha_3)$ and further mathematical manipulation,

$$(ds')^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} G_{ij} d\alpha_i d\alpha_j \tag{36}$$

Where,

$$G_{ij} = \left(g_{ii} + \sum_{k=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_k}\xi_k\right)\delta_{ij} + g_{ii}\frac{\partial \xi_i}{\partial \alpha_j} + g_{jj}\frac{\partial \xi_j}{\partial \alpha_i} \tag{37}$$

Where, δ_{ij} is Kronecker delta notation.

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{38}$$

And, i, j, & k refer to three principal coordinates.

Now for $i = j$, in general,

$$(ds)_{ii}^2 = g_{ii}(d\alpha_i)^2 \tag{40}$$

$$(ds')_{ii}^2 = G_{ii}(d\alpha_i)^2 \tag{41}$$

and, $i \neq j$,

$$(ds)_{ij}^2 = g_{ii}(d\alpha_i)^2 + g_{jj}(d\alpha_j)^2 \tag{42}$$

$$(ds')_{ij}^2 = G_{ii}(d\alpha_i)^2 + G_{jj}(d\alpha_j)^2 - 2G_{ij}d\alpha_i d\alpha_j \tag{43}$$

Therefore, the normal strains are,

$$\varepsilon_{ii} = \frac{(ds')_{ii} - (ds)_{ii}}{(ds)_{ii}} = \sqrt{\frac{G_{ii}}{g_{ii}}} - 1 = \sqrt{1 + \frac{G_{ii} - g_{ii}}{g_{ii}}} - 1 \tag{44}$$

Since,

$$\frac{G_{ii} - g_{ii}}{g_{ii}} \ll 1 \tag{45}$$

We have the expansion,

$$\sqrt{1 + \frac{G_{ii} - g_{ii}}{g_{ii}}} = 1 + \frac{1}{2}\frac{G_{ii} - g_{ii}}{g_{ii}} - \cdots \cdots \tag{46}$$

And,

$$\varepsilon_{ii} = \frac{1}{2}\frac{G_{ii} - g_{ii}}{g_{ii}} \tag{47}$$

Shear strains,

$$\varepsilon_{ij} = \frac{\pi}{2} - \theta_{ij} \tag{48}$$

θ_{ij} for $i = 1$ and $j = 2$ is shown in the figure below,
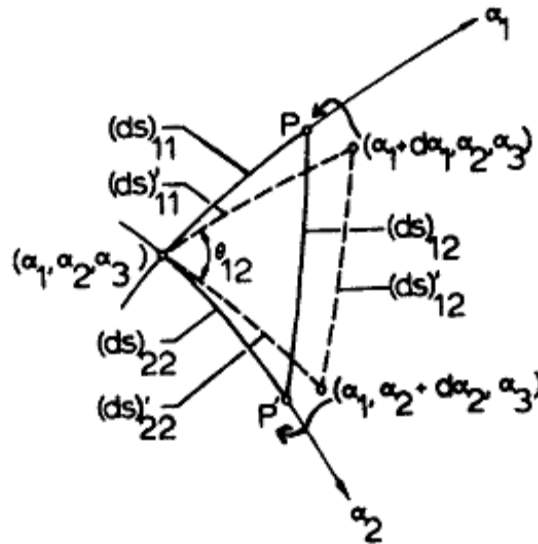
Fig 4. Shear deformation in the plane of the reference surface.

Using the cosine law, we may determine this angle,

$$(ds')_{ij}^2 = (ds')_{ii}^2 + (ds')_{jj}^2 - 2(ds')_{ii}(ds')_{jj}\cos\theta_{ij} \tag{49}$$

From equation (41), (43) and (49),

$$\cos\theta_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \tag{50}$$

$$\cos\left(\frac{\pi}{2} - \varepsilon_{ij}\right) = \sin\varepsilon_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \tag{51}$$

Since for reasonable shear strain magnitudes

$$\sin\varepsilon_{ij} \cong \varepsilon_{ij}$$

And,

$$\frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \cong \frac{G_{ij}}{\sqrt{g_{ii}g_{jj}}} \tag{52}$$

So, shear strain,

$$\varepsilon_{ij} = \frac{G_{ij}}{\sqrt{g_{ii}g_{jj}}} \tag{53}$$

Substituting equations (31) to (34) and (37) in eqn. (47) and for i=1, we finally get,

$$\varepsilon_{11} = \frac{1}{A_1(1 + \alpha_3/R_1)}\left(\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2} + U_3\frac{A_1}{R_1}\right) \tag{54}$$

Similarly, we can get,

$$\varepsilon_{22} = \frac{1}{A_2(1 + \alpha_3/R_2)}\left(\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1} + U_3\frac{A_2}{R_2}\right) \tag{55}$$

$$\varepsilon_{33} = \frac{\partial U_3}{\partial \alpha_3} \tag{56}$$

Again, Substituting equations (31) to (34) and (37) in eqn. (47) and for i=1, j=2,

$$\varepsilon_{12} = \frac{A_1(1 + \alpha_3/R_1)}{A_2(1 + \alpha_3/R_2)}\frac{\partial}{\partial \alpha_2}\left(\frac{U_1}{A_1(1 + \alpha_3/R_1)}\right)$$
$$+ \frac{A_2(1 + \alpha_3/R_2)}{A_1(1 + \alpha_3/R_1)}\frac{\partial}{\partial \alpha_1}\left(\frac{U_2}{A_2(1 + \alpha_3/R_2)}\right) \tag{57}$$

Similarly, we can get,

$$\varepsilon_{13} = A_1 \left(1 + \frac{\alpha_3}{R_1}\right) \frac{\partial}{\partial \alpha_3} \left(\frac{U_1}{A_1(1 + \alpha_3/R_1)}\right) + \frac{1}{A_1(1 + \alpha_3/R_1)} \frac{\partial U_3}{\partial \alpha_1} \tag{58}$$

$$\varepsilon_{23} = A_2 \left(1 + \frac{\alpha_3}{R_2}\right) \frac{\partial}{\partial \alpha_3} \left(\frac{U_2}{A_2(1 + \alpha_3/R_2)}\right) + \frac{1}{A_2(1 + \alpha_3/R_2)} \frac{\partial U_3}{\partial \alpha_2} \tag{59}$$

These equations give relationships among strain, displacement and the structure geometry.

LOVE SIMPLIFICATIONS

A E H Love had simplified the strain equations shown in the previous section (eqns. (54) to (59)) by assuming that for a thin walled shell the displacements in the α_1 and α_2 directions vary linearly through the shell thickness, where as displacements in the α_3 direction are independent of α_3.

$$U_1(\alpha_1, \alpha_2, \alpha_3) = u_1(\alpha_1, \alpha_2) + \alpha_3 \beta_1(\alpha_1, \alpha_2) \tag{60}$$

$$U_2(\alpha_1, \alpha_2, \alpha_3) = u_2(\alpha_1, \alpha_2) + \alpha_3 \beta_2(\alpha_1, \alpha_2) \tag{61}$$

$$U_3(\alpha_1, \alpha_2, \alpha_3) = u_3(\alpha_1, \alpha_2) \tag{62}$$

Where, β_1 and β_2 represents angles.
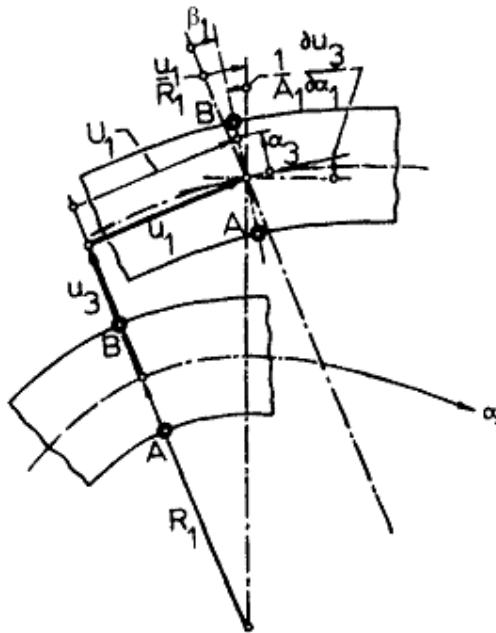
Fig 5. Illustration of the Love's assumption.

If we assume that we may neglect shear deflection, which implies that the normal shear strain deflection, which implies that the normal shear strains are 0,

$$\varepsilon_{13} = 0 \tag{63}$$

$$\varepsilon_{23} = 0 \tag{64}$$

Implying these assumptions (i.e. eqn. (60) to (64)) in the final strain equations of the previous section we get,

$$\varepsilon_{11} = \varepsilon^{\circ}{}_{11} + \alpha_3 k_{11} \tag{65}$$

$$\varepsilon_{22} = \varepsilon^{\circ}{}_{22} + \alpha_3 k_{22} \tag{66}$$

$$\varepsilon_{33} = 0 \tag{67}$$

$$\varepsilon_{12} = \varepsilon^{\circ}{}_{12} + \alpha_3 k_{12} \tag{68}$$

$$\varepsilon_{13} = 0 \tag{69}$$

$$\varepsilon_{23} = 0 \tag{70}$$

And,

$$\beta_1 = \frac{u_1}{R_1} - \frac{1}{A_1} \frac{\partial u_3}{\partial \alpha_1} \tag{71}$$

$$\beta_2 = \frac{u_2}{R_2} - \frac{1}{A_2} \frac{\partial u_3}{\partial \alpha_2} \tag{72}$$

Where, the membrane strains (independent of α_3) are,

$$\varepsilon^{\circ}{}_{11} = \frac{1}{A_1} \frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{u_3}{R_1} \tag{73}$$

$$\varepsilon^{\circ}{}_{22} = \frac{1}{A_2} \frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} + \frac{u_3}{R_2} \tag{74}$$

$$\varepsilon^{\circ}{}_{12} = \frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{u_2}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{u_1}{A_1} \right) \tag{75}$$

And where the bending strains (charge in curvature terms),

$$k_{11} = \frac{1}{A_1} \frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \tag{76}$$

$$k_{22} = \frac{1}{A_2} \frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \tag{77}$$

$$k_{12} = \frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{\beta_2}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{\beta_1}{A_1} \right) \tag{78}$$

MEMBRANE FORCES AND BENDING MOMENTS

Here we integrate all stresses acting on a shell element whose dimensions are infinitesimal in the α_1 and α_2 directions and equal to the shell thickness in normal direction. Solving eqns. (26) to (28) for stresses yields,

$$\sigma_{11} = \frac{E}{1 - \mu^2} (\varepsilon_{11} + \mu\varepsilon_{22}) \tag{79}$$

$$\sigma_{22} = \frac{E}{1 - \mu^2} (\varepsilon_{22} + \mu\varepsilon_{11}) \tag{80}$$

$$\sigma_{12} = \varepsilon_{12} G \tag{81}$$

Substituting eqns. (65), (66) and (68) gives,

$$\sigma_{11} = \frac{E}{1 - \mu^2} [\varepsilon°_{11} + \mu\varepsilon°_{22} + \alpha_3(k_{11} + \mu k_{22})] \tag{82}$$

$$\sigma_{22} = \frac{E}{1 - \mu^2} [\varepsilon°_{22} + \mu\varepsilon°_{11} + \alpha_3(k_{22} + \mu k_{11})] \tag{83}$$

$$\sigma_{12} = G(\varepsilon°_{12} + \alpha_3 k_{12}) \tag{84}$$
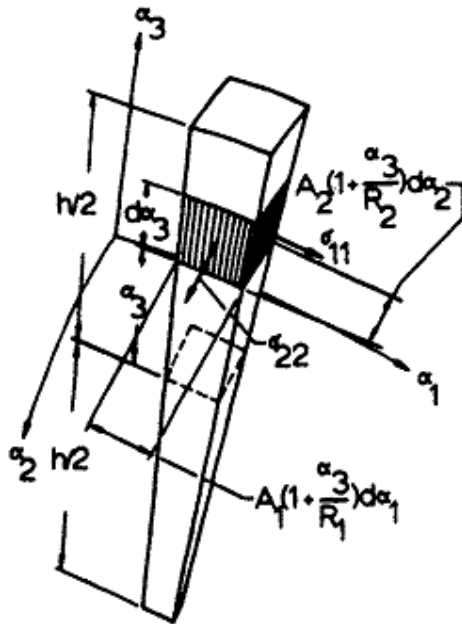
Fig 6. An element cut from a shell that is of infinitesimal cross-sectional dimensions, but extends through the entire thickness of the shell.

Referring to the figure above the force in the α_1 direction acting on a strip of the element face of height $d\alpha_3$ and width $A_2(1 + \alpha_3/R_2)d\alpha_2$ is

$$\sigma_{11}A_2\left(1 + \frac{\alpha_3}{R_2}\right)d\alpha_2 d\alpha_3$$

Thus the total force acting on the element in the α_1 direction is,

$$\int_{\alpha_3=-h/2}^{\alpha_3=h/2} \sigma_{11}A_2\left(1 + \frac{\alpha_3}{R_2}\right)d\alpha_2 d\alpha_3$$

and the force per unit length of neutral surface $A_2 d\alpha_2$ is

$$N_{11} = \int_{-h/2}^{h/2} \sigma_{11}\left(1 + \frac{\alpha_3}{R_2}\right)d\alpha_3 \tag{85}$$

Neglecting the second term in parentheses, we obtain

$$N_{11} = \int_{-h/2}^{h/2} \sigma_{11} d\alpha_3 \tag{86}$$

Substituting value of σ_{11} from eqn. (82) we get,

$$N_{11} = K(\varepsilon^\circ{}_{11} + \mu\varepsilon^\circ{}_{22}) \tag{87}$$

Where,

$$K = \frac{Eh}{1 - \mu^2} \tag{88}$$

Here, K is called the membrane stiffness. Similarly, integrating σ_{22} on the α_2 face of the element with the shear stresses $\sigma_{12} = \sigma_{21}$ gives,

$$N_{22} = K(\varepsilon^\circ{}_{22} + \mu\varepsilon^\circ{}_{11}) \tag{89}$$

$$N_{12} = N_{21} = \frac{K(1 - \mu)}{2}\varepsilon^\circ{}_{12} \tag{90}$$

To obtain bending moments, first the bending moment is expressed about the neutral surface due to element strip $A_2(1 + \alpha_3/R_2)d\alpha_2 d\alpha_3$,

$$\sigma_{11}\alpha_3 A_2(1 + \alpha_3/R_2)d\alpha_2 d\alpha_3$$

Thus the total bending moment acting on the element in the α_1 direction is

$$\int_{\alpha_3=-h/2}^{\alpha_3=h/2} \sigma_{11}\alpha_3 A_2(1 + \alpha_3/R_2)d\alpha_2 d\alpha_3$$

So, the bending moment per unit length of neutral surface is

$$M_{11} = \int_{-h/2}^{h/2} \sigma_{11}\alpha_3(1 + \alpha_3/R_2)d\alpha_3 \tag{91}$$

Neglecting the second term in parentheses, we get

$$M_{11} = \int_{-h/2}^{h/2} \sigma_{11} \alpha_3 d\alpha_3 \tag{92}$$

Again, substituting value of σ_{11} from eqn. (82) we get,

$$M_{11} = D(k_{11} + \mu k_{22}) \tag{93}$$

Where,

$$D = \frac{Eh^3}{12(1 - \mu^2)} \tag{94}$$

D is called the bending stiffness. Similarly integrating σ_{22} and $\sigma_{12} = \sigma_{21}$ we can obtain,

$$M_{22} = D(k_{22} + \mu k_{11}) \tag{95}$$

$$M_{12} = M_{21} = \frac{D(1 - \mu)}{2} k_{12} \tag{96}$$

While we have assumed that strains ε_{12} and ε_{23} due to transverse shear stresses σ_{13} and σ_{23} are negligible, we will never neglect the transverse shear forces.

$$Q_{13} = \int_{-h/2}^{h/2} \sigma_{13} d\alpha_3 \tag{97}$$

And,

$$Q_{23} = \int_{-h/2}^{h/2} \sigma_{23} d\alpha_3 \tag{98}$$

These forces will be defined by resulting equations themselves.

Now If we solve equation eqns. (87), (89), (90), (93), (95), and (96) for the strains, equations (82) to (84) become,

$$\sigma_{11} = \frac{N_{11}}{h} + \frac{12M_{11}}{h^3} \alpha_3 \tag{100}$$

$$\sigma_{22} = \frac{N_{22}}{h} + \frac{12M_{22}}{h^3} \alpha_3 \tag{101}$$

$$\sigma_{12} = \frac{N_{12}}{h} + \frac{12M_{12}}{h^3} \alpha_3 \tag{102}$$

Note that, it was assumed in this section that the reference surface is halfway between the inner and outer surfaces of the shell.

HAMILTON'S PRINCIPLE

Hamilton's principle is a minimization principle that seems to apply to all of mechanics and most classical physics. It is the end of a development that started in the second century B.C. with Hero of Alexandria, who stated that light always takes the shortest path. In 1834 Hamilton postulated that while there are usually several possible paths along which a dynamic system may move from one point to another in space and time, the path actually followed is the one that minimizes the time integral of the difference between the kinetic and potential energies. In terms of the calculus of variations, developed primarily by Euler and Bernoulli in the 18th century, it is usually stated is

$$\delta \int_{t_1}^{t_2} (K - U + W_{nc})\, dt = 0 , \qquad \delta \bar{r}_\iota = 0 , \qquad t = t_1, t_2 \tag{103}$$

Where $\delta \bar{r}_\iota$ are the variations of displacements (virtual displacements), K is the kinetic energy, U is the strain energy, W_{nc} is any additional energy input to the system, and δ is the variation, operationally equivalent to a total differential. In general, Hamilton's principle can be viewed as an axiom, replacing the axiom of Newton's second law for dynamic problems. In other words, we either accept Newton's second law as an axiom to derive Hamilton's principle from it for dynamic problems, or we accept Hamilton's principle as an axiom and derive Newton's second law from it.

LOVE'S EQUATIONS BY THE WAY OF HAMILTON'S PRINCIPLE

Hamilton's principle (multiplied here by –1 for convenience),

$$\delta \int_{t_1}^{t_2} (U - K - W_{in})\, dt = 0 \tag{104}$$

Where, W_{in} is the total input energy,

$$W_{in} = E_B + E_L \tag{105}$$

The times t_1 and t_0 are arbitrary, except that at $t = t_1$ and, $t = t_0$ all variations are 0. The symbol δ is the variational symbol and is treated mathematically like differential symbol. Variational displacements are arbitrary.

From eqns. (104) and (105), we obtain (taking the variational operator inside the integral),

$$\int_{t_1}^{t_2} (\delta U - \delta E_B - \delta E_L - \delta K)\, dt = 0 \tag{106}$$

Where,

δK =Variation in kinetic energy of one infinitesimal element,

δE_L =The variation of energy introduced into the shell by distributed load components

δE_B =Variation of energy due to boundary load

δU =Variation of strain energy stored in one infinitesimal element

Now, taking the time integral of these energy terms and subsequently substituting them in eqn. (106) results into a complex equation constituting of double and triple integrals. The equation can be satisfied only if each of the triple and double integral parts is 0 individually. Moreover, since the variational displacements are arbitrary, each integral equation can be satisfied only if the coefficients of the variational displacement are 0. Thus the coefficients of the triple integral set to zero give the following five equations:

$$-\frac{\partial(N_{11}A_2)}{\partial\alpha_1} - \frac{\partial(N_{21}A_1)}{\partial\alpha_2} - N_{12}\frac{\partial A_1}{\partial\alpha_2} + N_{22}\frac{\partial A_2}{\partial\alpha_1} - A_1A_2\frac{Q_{13}}{R_1} + A_1A_2\rho h\ddot{u}_1$$
$$= A_1A_2q_1 \tag{107}$$

$$-\frac{\partial(N_{12}A_2)}{\partial\alpha_1} - \frac{\partial(N_{22}A_1)}{\partial\alpha_2} - N_{12}\frac{\partial A_2}{\partial\alpha_1} + N_{11}\frac{\partial A_1}{\partial\alpha_2} - A_1A_2\frac{Q_{23}}{R_2} + A_1A_2\rho h\ddot{u}_2$$
$$= A_1A_2q_2 \tag{108}$$

$$-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1} - \frac{\partial(Q_{23}A_1)}{\partial\alpha_2} + A_1A_2\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\right) + A_1A_2\rho h\ddot{u}_3 = A_1A_2q_3 \tag{109}$$

Where, Q_{13} and Q_{23} are defined by,

$$\frac{\partial(M_{11}A_2)}{\partial\alpha_1} + \frac{\partial(M_{21}A_1)}{\partial\alpha_2} + M_{12}\frac{\partial A_1}{\partial\alpha_2} - M_{22}\frac{\partial A_2}{\partial\alpha_1} - Q_{13}A_1A_2 = 0 \tag{110}$$

$$\frac{\partial(M_{12}A_2)}{\partial\alpha_1} + \frac{\partial(M_{22}A_1)}{\partial\alpha_2} + M_{21}\frac{\partial A_2}{\partial\alpha_1} - M_{11}\frac{\partial A_1}{\partial\alpha_2} - Q_{23}A_1A_2 = 0 \tag{111}$$

These five equations are known as **Love's equations**. They define the motion (or static deflection, for all it matters) due to any type of pressure load. Shear deflection and rotary inertia are not included.

EQUATIONS OF MOTION

Equations of motion for circular cylindrical shell can be obtained from Love's equations. We have before that for the mentioned structure,

$$\alpha_1 = x, \alpha_2 = \theta; \qquad R_x = \infty, R_\theta = a \tag{112}$$

Fundamental form equation,

$$(ds)^2 = (dx)^2 + a^2(d\theta)^2 \tag{113}$$

And, Lamé parameters,

$$A_1 = 1, A_2 = a \tag{114}$$

Substituting eqns. (112) and (114) in Love's equations and neglecting transverse shear force, Q_{x3} and longitudinal force, N_{11} we can obtain the equations of motion for circular cylindrical shell, they are,

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{a}\frac{\partial N_{\theta x}}{\partial \theta} + q_x = \rho h \frac{\partial^2 u_x}{\partial t^2} \tag{115}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{a}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{Q_{\theta 3}}{a} + q_\theta = \rho h \frac{\partial^2 u_\theta}{\partial t^2} \tag{116}$$

$$\frac{\partial Q_{x3}}{\partial x} + \frac{1}{a}\frac{\partial Q_{\theta 3}}{\partial \theta} - \frac{N_{\theta\theta}}{a} + q_x = \rho h \frac{\partial^2 u_3}{\partial t^2} \tag{117}$$

Where,

$$Q_{x3} = \frac{\partial M_{xx}}{\partial x} + \frac{1}{a}\frac{\partial M_{\theta x}}{\partial \theta} \tag{118}$$

$$Q_{\theta 3} = \frac{\partial M_{x\theta}}{\partial x} + \frac{1}{a}\frac{\partial M_{\theta\theta}}{\partial \theta} \tag{119}$$

The strain-displacement relations become,

$$\varepsilon^\circ_{xx} = \frac{\partial u_x}{\partial x} \tag{120}$$

$$\varepsilon^\circ_{\theta\theta} = \frac{1}{a}\frac{\partial u_\theta}{\partial \theta} + \frac{u_3}{a} \tag{121}$$

$$\varepsilon^\circ_{x\theta} = \frac{\partial u_\theta}{\partial x} + \frac{1}{a}\frac{\partial u_x}{\partial \theta} \tag{122}$$

$$k_{xx} = \frac{\partial \beta_x}{\partial x} \tag{123}$$

$$k_{\theta\theta} = \frac{1}{a}\frac{\partial \beta_\theta}{\partial \theta} \tag{124}$$

$$k_{x\theta} = \frac{\partial \beta_\theta}{\partial x} + \frac{1}{a}\frac{\partial \beta_x}{\partial \theta} \tag{125}$$

And, β_1 and β_2 become,

$$\beta_x = -\frac{\partial u_3}{\partial x} \tag{126}$$

$$\beta_\theta = \frac{u_\theta}{a} - \frac{1}{a}\frac{\partial u_3}{\partial \theta} \tag{127}$$

NATURAL FREQUENCIES

SIMPLY SUPPORTED CIRCULAR CYLINDRICAL SHELL

For simply supported shell the boundary conditions are,

$$u_3(0,\theta,t) = 0 \tag{128}$$

$$u_\theta(0,\theta,t) = 0 \tag{129}$$

$$M_{xx}(0,\theta,t) = 0 \tag{130}$$

$$N_{xx}(0,\theta,t) = 0 \tag{131}$$

And,

$$u_3(L,\theta,t) = 0 \tag{132}$$

$$u_\theta(L,\theta,t) = 0 \tag{133}$$

$$M_{xx}(L,\theta,t) = 0 \tag{134}$$

$$N_{xx}(L,\theta,t) = 0 \tag{135}$$

Equations of motion for circular cylindrical shell can be obtained from eqns. (115) to (117), by setting $q_i = 0$, we get,

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{a}\frac{\partial N_{\theta x}}{\partial \theta} - \rho h \frac{\partial^2 u_x}{\partial t^2} = 0 \tag{136}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{a}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{Q_{\theta 3}}{a} - \rho h \frac{\partial^2 u_\theta}{\partial t^2} = 0 \tag{137}$$

$$\frac{\partial Q_{x3}}{\partial x} + \frac{1}{a}\frac{\partial Q_{\theta 3}}{\partial \theta} - \frac{N_{\theta\theta}}{a} - \rho h \frac{\partial^2 u_3}{\partial t^2} = 0 \tag{138}$$

With all terms defined by eqns. (118) to (127).

At a natural frequency every point in the elastic system moves harmonically, we may assume that,

$$u_x(x,\theta,t) = U_x(x,\theta)e^{j\omega t} \tag{139}$$

$$u_\theta(x, \theta, t) = U_\theta(x, \theta)e^{j\omega t} \tag{140}$$

$$u_3(x, \theta, t) = U_3(x, \theta)e^{j\omega t} \tag{141}$$

Substituting these into eqns. (136) to (138),

$$\frac{\partial N'_{xx}}{\partial x} + \frac{1}{a}\frac{\partial N'_{\theta x}}{\partial \theta} + \rho h\omega^2 U_x = 0 \tag{142}$$

$$\frac{\partial N'_{x\theta}}{\partial x} + \frac{1}{a}\frac{\partial N'_{\theta\theta}}{\partial \theta} + \frac{Q'_{\theta 3}}{a} + \rho h\omega^2 U_\theta = 0 \tag{143}$$

$$\frac{\partial Q'_{x3}}{\partial x} + \frac{1}{a}\frac{\partial Q'_{\theta 3}}{\partial \theta} - \frac{N'_{\theta\theta}}{a} + \rho h\omega^2 U_3 = 0 \tag{144}$$

Circular cylindrical shell is treated as combination of ring and simply supported beam and thus based on the experience of ring and simply supported beam, the following solutions are assumed

$$U_x(x, \theta) = A\cos\frac{m\pi x}{L}\cos n(\theta - \phi) \tag{145}$$

$$U_\theta(x, \theta) = B\sin\frac{m\pi x}{L}\sin n(\theta - \phi) \tag{146}$$

$$U_3(x, \theta) = C\sin\frac{m\pi x}{L}\cos n(\theta - \phi) \tag{147}$$

Utilizing these solutions the equations of motion ultimately become,

$$\left\{\rho h\omega^2 - K\left[\left(\frac{m\pi}{L}\right)^2 + \frac{1-\mu}{2}\left(\frac{n}{a}\right)^2\right]\right\}A + \left(K\frac{1+\mu}{2}\frac{n}{a}\frac{m\pi}{L}\right)B + \left(K\frac{\mu}{a}\frac{m\pi}{L}\right)C = 0 \tag{148}$$

$$\left(K\frac{1+\mu}{2}\frac{n}{a}\frac{m\pi}{L}\right)A + \left\{\rho h\omega^2 - \left(K + \frac{D}{a^2}\right)\left[\frac{1-\mu}{2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]\right\}B$$
$$+ \left\{-\frac{K}{a}\frac{n}{a} - \frac{D}{a}\frac{n}{a}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]\right\}C = 0 \tag{149}$$

$$\left(\frac{\mu K}{a}\frac{m\pi}{L}\right)A + \left\{-\frac{K}{a}\frac{n}{a}\right.$$
$$-\frac{D}{a}\frac{n}{a}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]\right\}B$$
$$+ \left\{\rho h\omega^2 - D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 - \frac{K}{a^2}\right\}C = 0 \tag{150}$$

Or,

$$\begin{bmatrix} \rho h\omega^2 - k_{11} & k_{12} & k_{13} \\ k_{21} & \rho h\omega^2 - k_{22} & k_{23} \\ k_{31} & k_{32} & \rho h\omega^2 - k_{33} \end{bmatrix}\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0 \tag{151}$$

Where,

$$k_{11} = K\left[\left(\frac{m\pi}{L}\right)^2 + \frac{1-\mu}{2}\left(\frac{n}{a}\right)^2\right] \tag{152}$$

$$k_{12} = k_{21} = K\frac{1+\mu}{2}\frac{n}{a}\frac{m\pi}{L} \tag{153}$$

$$k_{13} = k_{31} = \frac{\mu K}{a}\frac{m\pi}{L} \tag{154}$$

$$k_{22} = \left(K + \frac{D}{a^2}\right)\left[\frac{1-\mu}{2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right] \tag{155}$$

$$k_{23} = k_{32} = -\frac{K}{a}\frac{n}{a} - \frac{D}{a}\frac{n}{a}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right] \tag{156}$$

$$k_{33} = D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 + \frac{K}{a^2} \tag{157}$$

For a nontrivial solution, the determinant of equation (151) has to be 0. Expanding the determinant gives,

$$\omega^6 + a_1\omega^4 + a_2\omega^2 + a_3 = 0 \tag{158}$$

Where,

$$a_1 = -\frac{1}{\rho h}(k_{11} + k_{22} + k_{33}) \tag{159}$$

$$a_2 = \frac{1}{(\rho h)^2}\left(k_{11}k_{33} + k_{22}k_{33} + k_{11}k_{22} - k_{23}^2 - k_{12}^2 - k_{13}^2\right) \tag{160}$$

$$a_3 = \frac{1}{(\rho h)^3}\left(k_{11}k_{23}^2 + k_{22}k_{13}^2 + k_{33}k_{12}^2 + 2k_{12}k_{23}k_{13} - k_{11}k_{22}k_{33}\right) \tag{162}$$

In the above equations,

K = Membrane stiffness

D = Bending stiffness

L = Length of the shell

h = Thickness of the shell

a = Radius of the neutral membrane; $\left(\frac{ID}{2} + \frac{h}{2}\right)$

ρ = Density of material

$m = 1, 2, 3, ...$; Longitudinal parameter

$n = 0, 1, 2, 3, ...$; Circumferential parameter

The solutions of this equation are,

$$\omega_{1mn}^2 = -\frac{2}{3}\sqrt{a_1^2 - 3a_2}\cos\frac{\alpha}{3} - \frac{a_1}{3} \tag{163}$$

$$\omega_{2mn}^2 = -\frac{2}{3}\sqrt{a_1^2 - 3a_2}\cos\frac{\alpha + 2\pi}{3} - \frac{a_1}{3} \tag{164}$$

$$\omega_{3mn}^2 = -\frac{2}{3}\sqrt{a_1^2 - 3a_2}\cos\frac{\alpha + 4\pi}{3} - \frac{a_1}{3} \tag{165}$$

Where,

$$\alpha = \cos^{-1}\frac{27a_3 + 2a_1^3 - 9a_1a_2}{2\sqrt{(a_1^2 - 3a_2)^3}} \tag{166}$$

Eqn. (158) is a bicubic equation, therefore for every m, n combination, three frequencies are obtained. The lowest is associated with the mode where the transverse component dominates, while the other two are usually higher by an order of magnitude and are associated with the mode where the displacements in the tangent plane dominate. For every m, n combination, we therefore have three different combinations of A, B, and C. The three mode component ratios for every m, n combination are given by,

$$\frac{A_i}{C_i} = -\frac{k_{13}(\rho h\omega_{imn}^2 - k_{22}) - k_{12}k_{23}}{(\rho h\omega_{imn}^2 - k_{11})(\rho h\omega_{imn}^2 - k_{22}) - k_{12}^2} \tag{167}$$

$$\frac{B_i}{C_i} = -\frac{k_{23}(\rho h\omega_{imn}^2 - k_{11}) - k_{21}k_{13}}{(\rho h\omega_{imn}^2 - k_{11})(\rho h\omega_{imn}^2 - k_{22}) - k_{12}^2} \tag{168}$$

Where, $i = 1, 2, 3$.

These expressions are useful for forced vibration analysis.

We have used MATLAB to generate programs for calculating natural frequencies of our simply supported circular cylindrical shell,

MATLAB PROGRAME FOR DETERMINING NATURAL FREQUENCY

```matlab
E=input('Modulus of elasticity = ');
mu=input('Poissons ratio = ');
ro=input('Density = ');
L=input('Length = ');
OD=input('Outer Diameter = ');
ID=input('Inner Diameter = ');
h=(OD-ID)/2;
a=ID/2+h/2;
y=input('Highest value of m: ');
z=input('Highest value of n: ');
omgasq=zeros(y,z+1);
omga=zeros(y,z+1);
freq=zeros(y,z+1);
format long e
K=(E*h)/(1-mu^2);
D=(E*h^3)/(12*(1-mu^2));
for m=1:y
    for n=0:z
        k11 = K*((m*pi/L)^2+((1-mu)/2)*(n/a)^2);
        k12 = K*((1+mu)/2)*(m*pi/L)*(n/a);
        k21 = k12;
        k13 = (mu*K/a)*(m*pi/L);
        k31 = k13;
        k22 = (K+D/a^2)*(((1-mu)/2)*(m*pi/L)^2+(n/a)^2);
        k23 = -(K*n/a^2)-(D*n/a^2)*((m*pi/L)^2+(n/a)^2);
        k32 = k23;
        k33 = D*((m*pi/L)^2+(n/a)^2)^2+(K/a^2);
        a1 = -(1/(ro*h))*(k11+k22+k33);
        a2 = (1/(ro*h)^2)*(k11*k33+k22*k33+k11*k22-k23^2-
k12^2-k13^2);
        a3 =
(1/(ro*h)^3)*(k11*k23^2+k22*k13^2+k33*k12^2+2*k12*k23*k13
-k11*k22*k33);
        alpha = acos((27*a3+2*a1^3-
9*a1*a2)/(2*sqrt((a1^2-3*a2)^3)));
        omgasq(m,n+1) = -(2/3)*sqrt(a1^2-
3*a2)*cos(alpha/3)-(a1/3);
        omga(m,n+1) = sqrt(omgasq(m,n+1));
        freq(m,n+1) = sqrt(omgasq(m,n+1))/(2*pi);
    end
end
format bank
disp('Natural Frequencies : ')
disp(freq)
```

RESULTS

Natural Frequencies (in Hertz):

** Along column, circumferential parameter, n varies from 0 to 10 and along rows, the longitudinal parameter, m varies from 1 to 10 **

Columns 1 through 6

729.16	35.59	1617.30	4572.43	8765.94	14175.40
1458.32	136.82	1620.07	4575.06	8768.71	14178.25
2187.48	302.46	1626.11	4579.56	8773.35	14183.02
2916.64	528.33	1637.47	4586.10	8779.89	14189.70
3645.80	809.18	1656.86	4594.92	8788.36	14198.31
4374.96	1139.17	1687.41	4606.31	8798.83	14208.86
5104.12	1512.26	1732.41	4620.62	8811.36	14221.38
5833.28	1922.55	1794.94	4638.24	8826.02	14235.88
6562.44	2364.48	1877.57	4659.61	8842.90	14252.38
7291.60	2832.96	1982.06	4685.19	8862.10	14270.91

Columns 7 through 11

20794.14	28619.67	37650.91	47887.27	59328.42
20797.04	28622.61	37653.87	47890.24	59331.41
20801.89	28627.51	37658.80	47895.20	59336.38
20808.68	28634.37	37665.71	47902.14	59343.35
20817.41	28643.19	37674.59	47911.06	59352.30
20828.10	28653.98	37685.44	47921.97	59363.25
20840.74	28666.73	37698.28	47934.86	59376.18
20855.35	28681.45	37713.09	47949.74	59391.11
20871.93	28698.15	37729.88	47966.61	59408.03
20890.49	28716.83	37748.66	47985.46	59426.94

WHERE SPECIFICATIONS OF THE SIMPLY SUPPORTED CIRCULAR CYLINDRICAL SHELL MODEL USED:

Density= 7850 kg/m^3
Modulus of elasticity= 210e9 Pa
Poisson's ratio= 0.3
Length= 2.2 m
Outer Diameter= 59.5 mm
Inner Diameter= 55.5 mm

FORCED VIBRATION BY MODAL EXPANSION

In the previous section we have determined natural frequencies of circular cylindrical shell. For the engineers, the ultimate reason for this preoccupation is found in the study of the forced response of shells. For instance, knowing the eigenvalues or natural frequencies makes it possible to obtain the forced solution in terms of these eigenvalues. This approach is called spectral representation or modal expansion and dates back to Bernoulli's work (Bernoulli, 1755). Forces will be assumed to be independent of the motion of the shell. This is an admissible approximation for most engineering shell vibration cases.

MODAL PARTICIPATION FACTOR

Any disturbance excites the various natural modes of a shell in various amounts. The amount of participation of each mode in the total dynamic response is defined by the modal participation factor. This factor may turn out to be 0 for certain modes and may approach large values for others, depending on the nature of the excitation.

In a mathematical sense, the natural modes of a shell structure represent orthogonal vectors that satisfy the boundary conditions of the structure. This vector space can be used to represent any response of the structure. In cases of finite-degree-of-freedom systems, the vector space is of finite dimension and the number of vectors or natural modes is equal to the number of degrees of freedom. For continuous systems, such as shells, the number of degrees of freedom is infinite. This means that the general solution will be an infinite series,

$$u_i(\alpha_1, \alpha_2, t) = \sum_{k=1}^{\infty} \eta_k \left(\int \right) U_{ik}(\alpha_1, \alpha_2) \qquad (169)$$

Where, $i = 1, 2, 3$. The U_{ik} are the natural mode components in the three principal directions. The modal participation factors η_k are unknown and have to be determined in the following.

The Love equations are of the form

$$L_i\{u_1, u_2, u_3\} - \lambda \dot{u}_i - \rho h \ddot{u}_i = -q_i \qquad (170)$$

where λ is an equivalent viscous damping factor. The viscous damping term was introduced through the forcing term, replacing the original q_i by $q_i - \lambda \dot{u}_i$. Also, the damping factor is assumed to be the same in all three principal directions.

The operators L_i are defined, from Love's equation, as

$$L_1\{u_1, u_2, u_3\} = \frac{1}{A_1 A_2}\left[\frac{\partial(N_{11}A_2)}{\partial \alpha_1} + \frac{\partial(N_{21}A_1)}{\partial \alpha_2} + N_{12}\frac{\partial A_1}{\partial \alpha_2} - N_{22}\frac{\partial A_2}{\partial \alpha_1}\right.$$
$$\left. + A_1 A_2 \frac{Q_{13}}{R_1}\right] \tag{171}$$

$$L_2\{u_1, u_2, u_3\} = \frac{1}{A_1 A_2}\left[\frac{\partial(N_{12}A_2)}{\partial \alpha_1} + \frac{\partial(N_{22}A_1)}{\partial \alpha_2} + N_{21}\frac{\partial A_2}{\partial \alpha_1} - N_{11}\frac{\partial A_1}{\partial \alpha_2}\right.$$
$$\left. + A_1 A_2 \frac{Q_{23}}{R_2}\right] \tag{172}$$

$$L_3\{u_1, u_2, u_3\} = \frac{1}{A_1 A_2}\left[\frac{\partial(Q_{13}A_2)}{\partial \alpha_1} + \frac{\partial(Q_{23}A_1)}{\partial \alpha_2} - A_1 A_2\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\right)\right] \tag{173}$$

All of the simplifications discussed previously can be applied here. Eqn. (169) is general and will apply for all geometries and simplifications. Substituting eqn. (169) in eqn. (170) gives,

$$\sum_{k=1}^{\infty}\left[\eta_k L_i\{U_{1k}, U_{2k}, U_{3k}\} - \lambda\dot{\eta}_k U_{ik} - \rho h\ddot{\eta}_k U_{ik}\right] = -q_i \tag{174}$$

However, from our eigenvalue analysis, we know that,

$$L_i\{U_{1k}, U_{2k}, U_{3k}\} = -\rho h\omega_k^2 U_{ik} \tag{175}$$

Substituting this in eqn. gives,

$$\sum_{k=1}^{\infty}(\rho h\ddot{\eta}_k + \lambda\dot{\eta}_k + \rho h\omega_k^2\eta_k)U_{ik} = q_i \tag{176}$$

We know that the natural modes U_{ik} are orthogonal. As in a Fourier analysis, here advantage of the orthogonality of the sine and cosine functions is taken. Finally the form obtained is,

$$\ddot{\eta}_k + \frac{\lambda}{\rho h}\dot{\eta}_k + \omega_k^2\eta_k = F_k \tag{177}$$

Where,

$$F_k = \frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1}(q_1 U_{1k} + q_2 U_{2k} + q_3 U_{3k})A_1 A_2 d\alpha_1 d\alpha_2 \tag{178}$$

$$N_k = \int_{\alpha_2} \int_{\alpha_1} (U_{1k}^2 + U_{2k}^2 + U_{3k}^2) A_1 A_2 d\alpha_1 d\alpha_2 \tag{179}$$

Thus, if we take k terms of the modal expansion series as approximation to an infinite number, we have to solve the equation defining the modal participation factors k times. There is no principal difficulty connected with this. The forcing functions q_1, q_2, and q_3 have to be given and the mode components U_{1k}, U_{2k} and U_{3k} and the natural frequencies ω_k have to be known, either as direct functional or numerical theoretical solutions of the eigenvalue problem or as experimental data in functional or numerical form. The mass density per unit shell surface ρh is obviously also known and the damping factor λ has to be given or has to be estimated. For shells of nonuniform thickness, h has to be moved inside the integrals.

SOLUTION OF MODAL PARTICIPATION FACTOR

The modal participation factor equation is a simple oscillator equation. Thus we may interpret the forced vibration of shells by considering the shell as composed of simple oscillators, where each oscillator consists of the shell restricted to vibrate in one of its natural modes. All these oscillators respond simultaneously, and the total shell vibration is simply the result of addition (superposition) of all the individual vibrations.

The simple oscillator equation is solved by the Laplace transformation technique. The solutions for subcritical, critical, and supercritical damping are derived, even though only the first case is of real importance in shell vibration applications.

The modal participation factor equation can be written as

$$\ddot{\eta}_k + 2\xi_k \omega_k \dot{\eta}_k + \omega_k^2 \eta_k = F_k(t) \tag{180}$$

Where,

$$F_k(t) = \frac{\int_{\alpha_2} \int_{\alpha_1} (q_1 U_{1k} + q_2 U_{2k} + q_3 U_{3k}) A_1 A_2 d\alpha_1 d\alpha_2}{\rho h N_k} \tag{181}$$

$$\xi_k = \frac{\lambda}{2\rho h \omega_k} \tag{182}$$

Where, ξ_k is called the modal damping coefficient . It is analogous to the damping coefficient in the simple oscillator problem.

Taking the Laplace transformation of Eq. (180), it can be solved for the modal participation factor in the Laplace domain:

$$\eta_k(s) = \frac{F_k(s) + \eta_k(0)(s + 2\xi_k \omega_k) + \dot{\eta}_k(0)}{(s + \xi_k \omega_k)^2 + \omega_k^2(1 - \xi_k^2)} \tag{183}$$

The inverse transformation depends on whether the term $1 - \xi_k^2$ is positive, zero, or negative. The positive case, when $\xi_k < 1$, is the most common since it is very difficult to dampen shells more than that. It is called the "subcritical case". The critical case occurs when $\xi_k = 1$ and has no practical significance other than that it defines the damping that causes an initial modal displacement to decay in the fastest possible time without an oscillation. Supercritical damping $\xi_k > 1$ occurs only if a shell has such a high damping that it creeps back from an initial modal displacement without overshooting the equilibrium position.

For the subcritical case ($\xi_k < 1$), we define a real and positive number γ_k:

$$\gamma_k = \omega_k \sqrt{1 - \xi_k^2} \qquad (184)$$

The inverse Laplace transformation of Eq. (183) then gives

$$\eta_k(t) = e^{-\xi_k \omega_k t} \left\{ \eta_k(0) \cos \gamma_k t + [\eta_k(0)\xi_k \omega_k + \dot{\eta}_k(0)] \frac{\sin \gamma_k t}{\gamma_k} \right\}$$
$$+ \frac{1}{\gamma_k} \int_0^1 F_k(\tau) e^{-\xi_k \omega_k (t-\tau)} \ \sin \gamma_k (t - \tau) d\tau \qquad (185)$$

The solution is given in the form of the convolution integral since the forcing function $F_k(t)$ is at this point arbitrary. Once it is known, the convolution integral can be evaluated. It is also possible to take the inverse Laplace transformation of Eq. (183) with a known forcing function directly.

Vibrations caused by initial conditions will be oscillatory but will decay exponentially with time. The convolution integral, when evaluated for a specific forcing, will divide into a transient part and possibly a steady-state part if the forcing is periodic. The transient part will decay exponentially with time.

A special case of considerable technical interest is when damping is 0. The solution reduces to

$$\eta_k(t) = \eta_k(0) \cos \omega_k t + \dot{\eta}_k(0) \frac{\sin \omega_k t}{\omega_k} + \frac{1}{\omega_k} \int_0^t F_k(\tau) \sin \omega_k (t - \tau) d\tau \qquad (186)$$

Since most structures are very lightly damped, Eq. (186) in often used to get an approximate response since it is much simpler to use.

Next, let us investigate the supercritical case ($\xi_k > 1$). In this case, the value of $1 - \xi_k^2$ is negative. Defining a real and positive number ε_k ,

$$\varepsilon_k = \omega_k \sqrt{\xi_k^2 - 1} \qquad (187)$$

We obtain, taking the inverse Laplace transformation,

$$\eta_k(t) = e^{-\xi_k \omega_k t} \left\{ \eta_k(0) \cosh \varepsilon_k t + [\eta_k(0)\xi_k \omega_k + \dot{\eta}_k(0)]\frac{\sinh \varepsilon_k t}{\varepsilon_k} \right\}$$

$$+ \frac{1}{\varepsilon_k} \int_0^1 F_k(\tau) e^{-\xi_k \omega_k(t-\tau)} \; \sinh \varepsilon_k(t-\tau) d\tau \tag{188}$$

The vibrations caused by initial conditions are now non oscillatory, however, if the forcing is periodic, an oscillatory steady-state solution will still result.

As a special case, we obtain the critical damping solution $\xi_k = 1$ by reduction:

$$\eta_k(t) = e^{-\omega_k t} \{\eta_k(0) + [\eta_k(0)\omega_k + \dot{\eta}_k(0)]t\} + \int_0^t F_k(\tau) e^{-\omega_k(t-\tau)}(t-\tau)d\tau \tag{189}$$

STEADY-STATE HARMONIC RESPONSE

A technically important case occurs when the load on the shell varies harmonically with time and when the onset of vibrations (the transient part) is of no interest. Using a complex notation to get the response to both sine and cosine loading, we may write the load as

$$q_i(\alpha_1, \alpha_2, t) = q_i^*(\alpha_1, \alpha_2)e^{-j\omega t} \tag{190}$$

Using eqn. (180), it becomes,

$$\ddot{\eta}_k + 2\xi_k \omega_k \dot{\eta}_k + \omega_k^2 \eta_k = F_k^* e^{j\omega t} \tag{191}$$

Where,

$$F_k^* = \frac{1}{\rho h N_k} \int_{\alpha_2} \int_{\alpha_1} (q_1^* U_{1k} + q_2^* U_{2k} + q_3^* U_{3k}) A_1 A_2 \, d\alpha_1 d\alpha_2 \tag{192}$$

At steady state, the response will be harmonic also but lagging behind by a phase angle ϕ_k,

$$\eta_k = A_k e^{j(\omega t - \phi_k)} \tag{193}$$

Substituting this gives,

$$A_k e^{-j\phi_k} = \frac{F_k^*}{\left(\omega_k^2 - \omega^2\right) + 2j\xi_k \omega_k \omega} \tag{194}$$

The magnitude of response is, therefore,

$$A_k = \frac{F_k^*}{\omega_k^2 \sqrt{[1 - (\omega/\omega_k)^2]^2 + 4\xi_k^2(\omega/\omega_k)^2}} \tag{195}$$

The phase lag is,

$$\phi_k = \tan^{-1} \frac{2\xi_k(\omega/\omega_k)}{1 - (\omega/\omega_k)^2} \tag{196}$$

As expected, a shell will behave similarly to a collection of simple oscillators. Whenever the excitation frequency coincides with one of the natural frequencies, a peak in the response curve will occur. The harmonic response solution is the same for subcritical and supercritical damping, except that for equal forcing, the response amplitudes at resonance become less and less pronounced as damping is increased until they are indistinguishable from the off-resonance response.

STEADY-STATE CIRCULAR CYLINDRICAL SHELL RESPONSE TO HARMONIC POINT LOAD WITH ALL MODE COMPONENTS CONSIDERED

For simply supported boundary conditions, there are two sets of natural modes to consider. For $\phi = 0, n = 0, 1, 2, 3, \ldots \ldots, m = 1, 2, 3, \ldots \ldots \ldots$, we have,

$$U_{xmni(1)} = \frac{A_{mni}}{C_{mni}} \cos(\frac{m\pi x}{L}) \cos n\theta \tag{197}$$

$$U_{\theta mni(1)} = \frac{B_{mni}}{C_{mni}} \sin\frac{m\pi x}{L} \sin n\theta \tag{198}$$

$$U_{3mni(1)} = \sin\frac{m\pi x}{L} \cos n\theta \tag{199}$$

and for $\phi = \pi/(2n)$ the set of natural modes, which is orthogonal to the modes of eqns. (197)–(199), is

$$U_{xmni(2)} = \frac{A_{mni}}{C_{mni}} \cos(\frac{m\pi x}{L}) \sin n\theta \tag{200}$$

$$U_{\theta mni(2)} = -\frac{B_{mni}}{C_{mni}} \sin\frac{m\pi x}{L} \cos n\theta \tag{201}$$

$$U_{3mni(2)} = \sin\frac{m\pi x}{L} \sin n\theta \tag{202}$$

A_{mni}/C_{mni} and B_{mni}/C_{mni} are obtained from eqns. (167) and (168).

Note that $i = 1, 2, 3$, corresponding to each of the natural frequencies for a given (m, n) combination. In most engineering applications, the natural frequencies associated with $i = 2$ and 3 are so high that the contribution of these modes can be neglected. But we consider them here.

The force shown in fig. 7 results in following loading description,

$$q_x(x, \theta, t) = 0 \tag{203}$$

$$q_\theta(x, \theta, t) = 0 \tag{204}$$

$$q_3(x, \theta, t) = F_3 e^{j\omega t} \frac{1}{a} \delta(\theta - \theta^*) \delta(x - x^*) \tag{205}$$

Where, imaginary part of $e^{j\omega t}$ represents $\sin \omega t$.
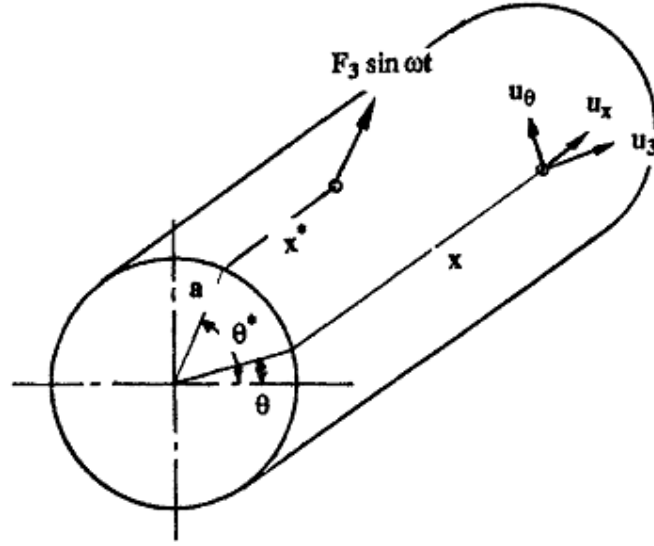
Fig. 7 Circular cylindrical shell acted on by a point force

For each set of natural modes solution i.e. displacement of a point on the shell surface in three principle coordinate directions is determined and then they are added together for the complete solution.

The final solutions for displacement of a point on the shell surface in three principle coordinate directions are,

$$u_x$$
$$= \sum_{i=1}^{3} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{F_3(\frac{A_{mni}}{C_{mni}}) \sin(\frac{m\pi x^*}{L}) \cos(\frac{m\pi x}{L}) \cos n(\theta - \theta^*) \sin(\omega t - \phi_{mni})}{\rho h N_{mni} f(\omega)} \tag{206}$$

$$u_\theta$$
$$= \sum_{i=1}^{3} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{F_3(\frac{B_{mni}}{C_{mni}}) \sin(\frac{m\pi x^*}{L}) \sin(\frac{m\pi x}{L}) \sin n(\theta - \theta^*) \sin(\omega t - \phi_{mni})}{\rho h N_{mni} f(\omega)} \tag{207}$$

$$u_3$$
$$= \sum_{i=1}^{3} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{F_3 \sin(\frac{m\pi x^*}{L}) \sin(\frac{m\pi x}{L}) \cos n(\theta - \theta^*) \sin(\omega t - \phi_{mni})}{\rho h N_{mni} f(\omega)} \tag{208}$$

Where,

$$N_{mni} \begin{cases} [\left(\frac{A_{mni}}{C_{mni}}\right)^2 + \left(\frac{B_{mni}}{C_{mni}}\right)^2 + 1]\frac{La\pi}{2} & if \ n \neq 0 \\ [\left(\frac{A_{mni}}{C_{mni}}\right)^2 + 1]La\pi & ; \ if \ n = 0 \end{cases}$$

(**209**)

And,

$$f(\omega) = \omega_{mni}^2 \sqrt{\left[1 - \left(\frac{\omega}{\omega_{mni}}\right)^2\right]^2 + 4\xi_{mni}^2 \left(\frac{\omega}{\omega_{mni}}\right)^2}$$

(**210**)

ϕ_{mni} is determined from the eqn. (196).

To determine the response of our simply supported circular cylindrical shell being subjected to a sinusoidal actuation based on the theories described in the preceding sections, we have written programs on MATLAB which given in the following section.

MATLAB PROGRAMS

To obtain motion of any specific point on the shell surface, while the shell is being excited by a sinusoidal force of constant amplitude and frequency, in terms of displacement (in α_3 direction) against time the following program has been used

```
disp('INPUT ALL DATAS IN SI UNIT')
E=input('Modulus of elasticity = ');
mu=input('Poissons ratio = ');
ro=input('Density = ');
L=input('Length = ');
OD=input('Outer Diameter = ');
ID=input('Inner Diameter = ');
h=(OD-ID)/2;
a=ID/2+h/2;
y=input('Highest value of m: ');
z=input('Highest value of n: ');
P3=input('Enter maximum value of force : ');
f=input('Excitation frequency (in Hertz) : ');
lmda=input('Equivalent viscous damping factor (in Ns/m) :
');
u3=0;
disp('Enter location of harmonic load')
loadx=input('x = ');
loadth=input('Theta (in radians) = ');
```

```matlab
disp('Enter location of the point of investigation')
x=input('X = ');
th=input('Theta (in radians) = ');
omgaextn=f*2*pi; t=1:0.01:10;
omgasq=zeros(y,z+1,3);
omga=zeros(y,z+1,3);
freq=zeros(y,z+1,3);
AbyC=zeros(y,z+1,3);
BbyC=zeros(y,z+1,3);
N=zeros(y,z+1,3);
format long e
K=(E*h)/(1-mu^2);
D=(E*h^3)/(12*(1-mu^2));
for i=1:3
    for m=1:y
        for n=0:z
        k11 = K*((m*pi/L)^2+((1-mu)/2)*(n/a)^2);
        k12 = K*((1+mu)/2)*(m*pi/L)*(n/a);
        k21 = k12;
        k13 = (mu*K/a)*(m*pi/L);
        k31 = k13;
        k22 = (K+D/a^2)*(((1-mu)/2)*(m*pi/L)^2+(n/a)^2);
        k23 = -(K*n/a^2)-(D*n/a^2)*((m*pi/L)^2+(n/a)^2);
        k32 = k23;
        k33 = D*((m*pi/L)^2+(n/a)^2)^2+(K/a^2);
        a1 = -(1/(ro*h))*(k11+k22+k33);
        a2 = (1/(ro*h)^2)*(k11*k33+k22*k33+k11*k22-k23^2-
k12^2-k13^2);
        a3 =
(1/(ro*h)^3)*(k11*k23^2+k22*k13^2+k33*k12^2+2*k12*k23*k13
-k11*k22*k33);
        alpha = acos((27*a3+2*a1^3-
9*a1*a2)/(2*sqrt((a1^2-3*a2)^3)));
        if i==1
        omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos(alpha/3)-(a1/3);
        elseif i==2
            omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos((alpha+2*pi)/3)-(a1/3);
        else
            omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos((alpha+4*pi)/3)-(a1/3);
        end

        omga(m,n+1,i) = sqrt(omgasq(m,n+1,i));
        freq(m,n+1,i) = sqrt(omgasq(m,n+1,i))/(2*pi);
        AbyC(m,n+1,i) = -(k13*(ro*h*(omga(m,n+1,i))^2-
k22)-k12*k23)/((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2);
```

```
        BbyC(m,n+1,i) = -(k23*(ro*h*(omga(m,n+1,i))^2-
k11)-k21*k13)/((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2);
        if ((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2)==0
            nom=0; denom=1;
        else
            zeta=lmda/(2*ro*h*omga(m,n+1,i));
        phi=atan(2*zeta*(omgaextn/omga(m,n+1,i))/(1-
(omgaextn/omga(m,n+1,i))^2));
        nom=P3*sin(m*pi*loadx/L)*sin(m*pi*x/L)*cos(n*(th-
loadth))*sin(omgaextn*t-phi);
        if n==0
            N(m,n+1,i)=((AbyC(m,n+1,i))^2+1)*L*a*pi;
        else

N(m,n+1,i)=((AbyC(m,n+1,i))^2+(BbyC(m,n+1,i))^2+1)*L*a*pi
/2;
        end
        denom =ro*h*N(m,n+1,i)*(omga(m,n+1,i))^2*sqrt((1-
(omgaextn/omga(m,n+1,i))^2)^2+4*zeta^2*(omgaextn/omga(m,n
+1,i))^2);
        end
        u3=u3+nom/denom;
        end
    end
end
plot(t,u3),grid
```

RESULTS

Motion of the point on the shell where force applied:

The following input parameters were defined:

Modulus of elasticity = 210e9 Pa

Poisson's ratio = 0.3

Density = 7850 kg/m^3

Length = 2.2 m

Outer Diameter = .0595 m

Inner Diameter = .0555 m

Highest value of m: 100

Highest value of n: 100

Enter maximum value of force : 10 N

Excitation frequency (in Hertz) : 36

Equivalent viscous damping factor (in Ns/m) : 175

Location of harmonic load,

x = 1.1 m

Theta (in radians) = pi

Location of the point of investigation,

X = 1.1 m

Theta (in radians) = pi

Then, for the amplitudes of displacement of points on a line along the length of the circular cylindrical shell at any given angle (i.e. α_2 coordinate), following program has been used,

```
disp('INPUT ALL VALUES IN SI UNIT')
E=input('Modulus of elasticity = ');
mu=input('Poissons ratio = ');
ro=input('Density = ');
L=input('Length (in meters)= ');
OD=input('Outer Diameter (in meters)= ');
ID=input('Inner Diameter (in meters)= ');
h=(OD-ID)/2;
a=ID/2+h/2;
y=input('Highest value of m: ');
```

```matlab
z=input('Highest value of n: ');
P3=input('Enter maximum value of force : ');
f=input('Excitation frequency (in Hertz) : ');
lmda=input('Equivalent viscous damping factor (in Ns/m) :
');
u3=zeros(1,L/0.01+1);
disp('Enter location of harmonic load')
loadx=input('x (in meters)= ');
loadth=input('Theta (in radians) = ');
disp('Enter the angle at which displacement along the
shell length is required')
x=0:0.01:L;
th=input('Theta (in radians) = ');
omgaextn=f*2*pi;
omgasq=zeros(y,z+1,3);
omga=zeros(y,z+1,3);
freq=zeros(y,z+1,3);
AbyC=zeros(y,z+1,3);
BbyC=zeros(y,z+1,3);
N=zeros(y,z+1,3);
format long e
K=(E*h)/(1-mu^2);
D=(E*h^3)/(12*(1-mu^2));
for i=1:3
    for m=1:y
        for n=0:z
        k11 = K*((m*pi/L)^2+((1-mu)/2)*(n/a)^2);
        k12 = K*((1+mu)/2)*(m*pi/L)*(n/a);
        k21 = k12;
        k13 = (mu*K/a)*(m*pi/L);
        k31 = k13;
        k22 = (K+D/a^2)*(((1-mu)/2)*(m*pi/L)^2+(n/a)^2);
        k23 = -(K*n/a^2)-(D*n/a^2)*((m*pi/L)^2+(n/a)^2);
        k32 = k23;
        k33 = D*((m*pi/L)^2+(n/a)^2)^2+(K/a^2);
        a1 = -(1/(ro*h))*(k11+k22+k33);
        a2 = (1/(ro*h)^2)*(k11*k33+k22*k33+k11*k22-k23^2-
k12^2-k13^2);
        a3 =
(1/(ro*h)^3)*(k11*k23^2+k22*k13^2+k33*k12^2+2*k12*k23*k13
-k11*k22*k33);
        alpha = acos((27*a3+2*a1^3-
9*a1*a2)/(2*sqrt((a1^2-3*a2)^3)));
        if i==1
        omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos(alpha/3)-(a1/3);
        elseif i==2
            omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos((alpha+2*pi)/3)-(a1/3);
        else
```

```
            omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos((alpha+4*pi)/3)-(a1/3);
        end

        omga(m,n+1,i) = sqrt(omgasq(m,n+1,i));
        freq(m,n+1,i) = sqrt(omgasq(m,n+1,i))/(2*pi);
        AbyC(m,n+1,i) = -(k13*(ro*h*(omga(m,n+1,i))^2-
k22)-k12*k23)/((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2);
        BbyC(m,n+1,i) = -(k23*(ro*h*(omga(m,n+1,i))^2-
k11)-k21*k13)/((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2);
        if ((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2)==0
            nom=0; denom=1;
        else
            zeta=lmda/(2*ro*h*omga(m,n+1,i));
        phi=atan(2*zeta*(omgaextn/omga(m,n+1,i))/(1-
(omgaextn/omga(m,n+1,i))^2));
        nom=P3*sin(m*pi*loadx/L)*sin(m*pi*x/L)*cos(n*(th-
loadth));
        if n==0
            N(m,n+1,i)=((AbyC(m,n+1,i))^2+1)*L*a*pi;
        else

N(m,n+1,i)=((AbyC(m,n+1,i))^2+(BbyC(m,n+1,i))^2+1)*L*a*pi
/2;
        end
        denom =ro*h*N(m,n+1,i)*(omga(m,n+1,i))^2*sqrt((1-
(omgaextn/omga(m,n+1,i))^2)^2+4*zeta^2*(omgaextn/omga(m,n
+1,i))^2);
        end
        u3=u3+nom./denom;
        end
    end
end
plot(x,u3),grid
```
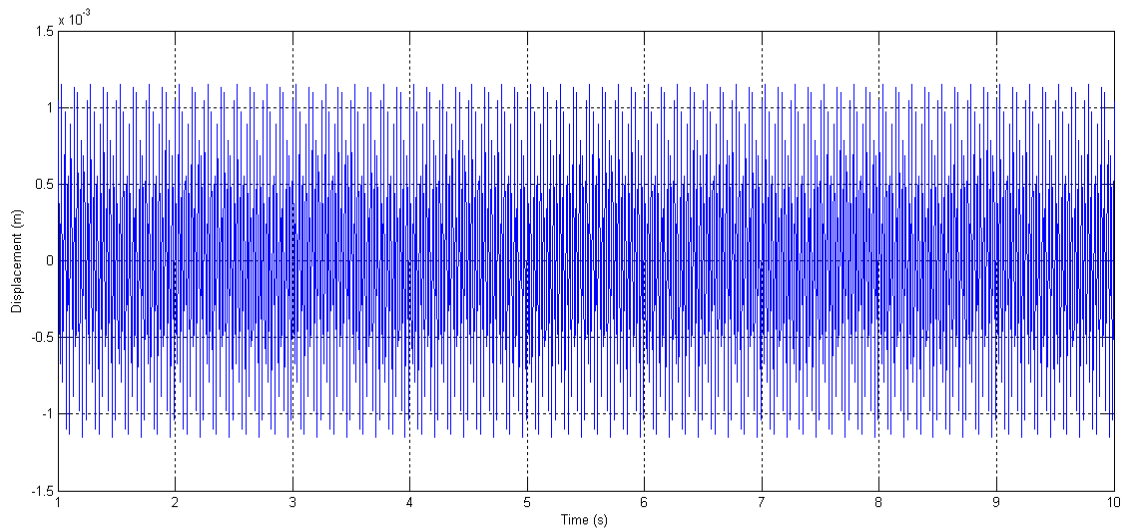
RESULTS
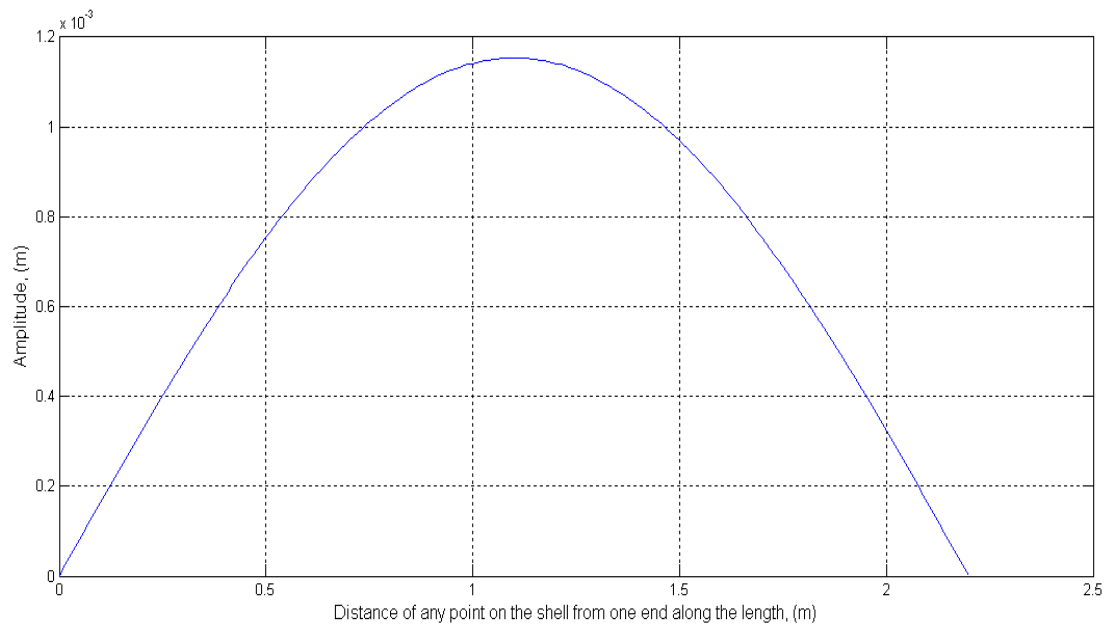
The following input parameters were defined:

Modulus of elasticity = 210e9 Pa

Poisson's ratio = 0.3

Density = 7850 kg/m^3

Length = 2.2 m

Outer Diameter = .0595 m

Inner Diameter = .0555 m

Highest value of m: 100

Highest value of n: 100

Enter maximum value of force: 10 N

Excitation frequency (in Hertz): 36

Equivalent viscous damping factor (in Ns/m): 175

Location of harmonic load,

x = 1.1 m

Theta (in radians) = pi

Enter the angle at which displacement along the shell length is required,

Theta (in radians) = pi

And, for resonance curve the following program is used,

```matlab
disp('INPUT ALL DATAS IN SI UNIT')
E=input('Modulus of elasticity = ');
mu=input('Poissons ratio = ');
ro=input('Density = ');
L=input('Length = ');
OD=input('Outer Diameter = ');
ID=input('Inner Diameter = ');
h=(OD-ID)/2;
a=ID/2+h/2;
y=input('Highest value of m: ');
z=input('Highest value of n: ');
P3=input('Enter maximum value of force : ');
disp('Enter frequency range (must be integer values,in
Hertz)')
f1=input('Start (1Hz or higher)= '); f2=input('End= ');
lmda=input('Equivalent viscous damping factor (in Ns/m) :
');
u3s=zeros(1,f2-f1+1);
disp('Enter location of harmonic load')
loadx=input('x = ');
loadth=input('Theta (in radians) = ');
disp('Enter location of the point of investigation')
x=input('X = ');
th=input('Theta (in radians) = ');
format long e
K=(E*h)/(1-mu^2);
D=(E*h^3)/(12*(1-mu^2));
for f=f1:f2
    u3=0; omgaextn=f*2*pi;
    omgasq=zeros(y,z+1,3);
omga=zeros(y,z+1,3);
freq=zeros(y,z+1,3);
AbyC=zeros(y,z+1,3);
BbyC=zeros(y,z+1,3);
N=zeros(y,z+1,3);

for i=1:3
    for m=1:y
        for n=0:z
        k11 = K*((m*pi/L)^2+((1-mu)/2)*(n/a)^2);
        k12 = K*((1+mu)/2)*(m*pi/L)*(n/a);
        k21 = k12;
```

```
k13 = (mu*K/a)*(m*pi/L);
k31 = k13;
k22 = (K+D/a^2)*(((1-mu)/2)*(m*pi/L)^2+(n/a)^2);
k23 = -(K*n/a^2)-(D*n/a^2)*((m*pi/L)^2+(n/a)^2);
k32 = k23;
k33 = D*((m*pi/L)^2+(n/a)^2)^2+(K/a^2);
a1 = -(1/(ro*h))*(k11+k22+k33);
a2 = (1/(ro*h)^2)*(k11*k33+k22*k33+k11*k22-k23^2-
k12^2-k13^2);
a3 =
(1/(ro*h)^3)*(k11*k23^2+k22*k13^2+k33*k12^2+2*k12*k23*k13
-k11*k22*k33);
alpha = acos((27*a3+2*a1^3-
9*a1*a2)/(2*sqrt((a1^2-3*a2)^3)));
if i==1
omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos(alpha/3)-(a1/3);
elseif i==2
omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos((alpha+2*pi)/3)-(a1/3);
else
omgasq(m,n+1,i) = -(2/3)*sqrt(a1^2-
3*a2)*cos((alpha+4*pi)/3)-(a1/3);
end

omga(m,n+1,i) = sqrt(omgasq(m,n+1,i));
freq(m,n+1,i) = sqrt(omgasq(m,n+1,i))/(2*pi);
AbyC(m,n+1,i) = -(k13*(ro*h*(omga(m,n+1,i))^2-
k22)-k12*k23)/((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2);
BbyC(m,n+1,i) = -(k23*(ro*h*(omga(m,n+1,i))^2-
k11)-k21*k13)/((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2);
if ((ro*h*(omga(m,n+1,i))^2-
k11)*(ro*h*(omga(m,n+1,i))^2-k22)-k12^2)==0
nom=0; denom=1;
else
zeta=lmda/(2*ro*h*omga(m,n+1,i));
phi=atan(2*zeta*(omgaextn/omga(m,n+1,i))/(1-
(omgaextn/omga(m,n+1,i))^2));
nom=P3*sin(m*pi*loadx/L)*sin(m*pi*x/L)*cos(n*(th-
loadth));
if n==0
N(m,n+1,i)=((AbyC(m,n+1,i))^2+1)*L*a*pi;
else

N(m,n+1,i)=((AbyC(m,n+1,i))^2+(BbyC(m,n+1,i))^2+1)*L*a*pi
/2;
end
```

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```
        denom =ro*h*N(m,n+1,i)*(omga(m,n+1,i))^2*sqrt((1-
(omgaextn/omga(m,n+1,i))^2)^2+4*zeta^2*(omgaextn/omga(m,n
+1,i))^2);
        end
        u3=u3+nom/denom;
        end
    end
end
u3s(1,f)=u3;
end
f=f1:f2;
plot(f,u3s),grid
```

RESULTS

Resonance curve for the point on the circular cylindrical shell where force is applied,
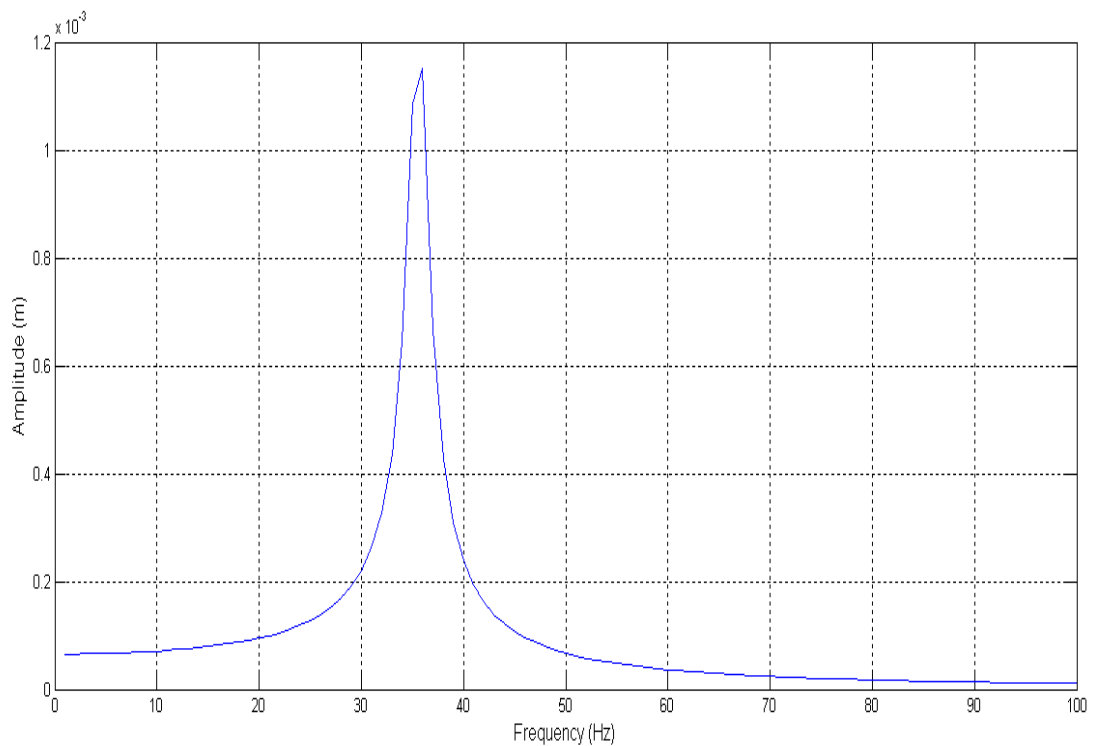
The following input parameters were defined:

Modulus of elasticity = 210e9 Pa

Poisson's ratio = 0.3

Density = 7850 kg/m^3

Length = 2.2 m

Outer Diameter = .0595 m

Inner Diameter = .0555 m

Highest value of m: 100

Highest value of n: 100

Maximum value of force: 10 N

Frequency range (must be integer values, in Hertz)

Start (1Hz or higher) = 1 Hz

End= 100 Hz

Equivalent viscous damping factor (in Ns/m) : 175

Location of harmonic load,

x = 1.1 m

Theta (in radians) = pi

Location of the point of investigation,

X = 1.1 m

Theta (in radians) = pi

CHAPTER 5

EXPERIMENTAL WORK

EQUIPMENTS:

1. Cylindrical pipe
2. Supports (for providing simply support)
3. Proximity sensor
4. Aluminum foils
5. Hammer
6. Oscilloscope
7. Cements, rods, sands etc for making concrete base

PROBLEM SPECIFICATION:

MATERIAL FOR SHELL:

 MILD STEEL

PROPERTIES OF MILD STEEL:

MODULUS OF RIGIDITY: 210 GPa

DENSITY: 7850 kg/m^3

POISSON RATIO: 0.3

MEASUREMENTS OF THE PIPE:

LENGTH: 2.59 m (8.5 feet)

RADIUS: 59.5 mm

THICKNESS: 2 mm

DISTANCE BETWEEN 2 SUPPORTS: 2.2 m

DESCRIPTION OF THE PROXIMITY SENSOR

Approvals and Safety Considerations

 The ECL202/ECL202e is compliant with the following CE directives:

Safety: 61010-1:2001

EMC: 61326-1, 61326-2-3

To maintain compliance with these standards, the following operating conditions must be maintained:

- All I/O connecting cables must be less than three meters in length
- AC power cables must be rated at a minimum of 250 V and 5 A
- AC power must be connected to a grounded mains outlet rated less than 20 A
- Power supply must have CE certification and provide safety isolation from the mains according to IEC60950 or 61010.
- Sensors must not be attached to parts operating at hazardous voltages in excess of 30 VRMS or 60 VDC
- All external connections must be SELV (Safety Extra Low Voltage).
- Use of the equipment in any other manner may impair the safety and EMI protections of the equipment.

DESCRIPTION

The Lion Precision ECL202 Eddy-Current Displacement Sensor provides high-resolution, noncontact measurement of position changes of a conductive target. The system consists of driver electronics and a probe calibrated for a specific material and range. The calibration information is detailed on a calibration certificate which is shipped with the system. The ECL202 provides a linear analog voltage proportional to changes in the target position and a digital switched (setpoint) output with a user programmed switching setpoint.

QUICK START INSTRUCTIONS

1. Connect the probe to the ECL202. The ECL202 is calibrated to a specific probe identified by serial number. The probe serial number must match the "USE PROBE S/N" label on the front of the ECL202.

2. Connect the output to a monitoring device.

3. Connect then apply power.

4. Adjust the probe position so the Range Indicator shows green

FRONT PANEL CONTROLS AND INDICATORS

LED Range Indicator

The Range Indicator monitors and displays the probe position within its calibrated range. The graphic below shows the indicator condition at various points within the calibrated range.
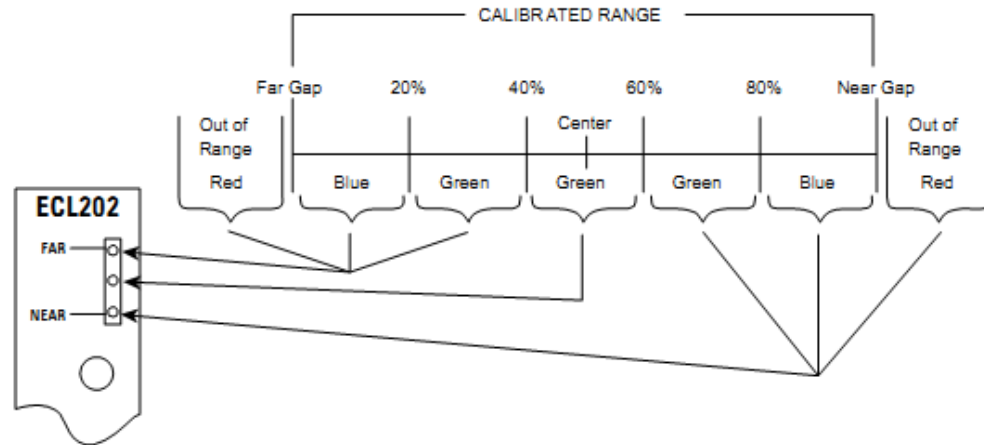
Figure: LED RANGE INDICATOR

The LED Range Indicator is independent of the output voltage and not affected by the Offset button. Shifting the output voltage by using the Offset button may allow an apparently valid output voltage to exist while the probe is out of range. When the Near or Far LED is red, the probe is out of range and the output voltage is not a reliable indication of the target position.

Offset Button

Pushing the Offset button shifts the DC level of the output voltage to the center of the voltage range (i.e. 5 V for a 0-10 V output). The button will only function when the probe is in the center 20% of its calibrated range (center green LED). If the center green LED on the Range Indicator is not on, the Offset button will not function. This function establishes a repeatable master point for reference measurements.

1. Place good part in the measurement area.

2. Position probe to center 20% of range (center indicator LED).

3. Press Offset button.

4. All subsequent measurements indicate deviation from center of range (5 V).

Resetting Offset

Hold the Offset button for four seconds to remove any output DC shift.

Set point Button

The ECL202 provides an adjustable set point at which a switched output activates. The output switch closes when the output voltage is more positive (larger gap) than the user-adjusted set point. Pressing the Set point button will set the threshold voltage to the current output voltage. The set point includes a 0.085V hysteresis, requiring that the sensor output drop 0.085V below the set point voltage before the switched output opens.

Analog Output Signal

The output signal is an analog voltage of 0-10 VDC. The output voltage is proportional to the probe-target gap. As the probe-target gap increases, the voltage becomes more positive. See the included calibration certificate for specific information.

Interpreting the Output Voltage

Output voltage change for a given change in the probe-target gap is called sensitivity. The sensitivity of the sensor is listed on the calibration certificate.

Change in gap calculation:

Gap Change = Voltage Change / Sensitivity

For example: With a sensitivity of 1V/2 µm and a voltage change of

+3 V, the probe-target gap has increased by 6 µm.

Remote Offset and Setpoint

The front panel Offset and Set point buttons can be activated remotely. Each remote input connects to an optoisolator. The functions are activated by applying 15-24 V to the remote control input terminals.

Note: Because the remote operation mimics front panel operation, activating the Offset function for more than four seconds will restore factory default value for Offset.

Setpoint Switch Output

When the output voltage is more positive than the user adjusted setpoint voltage, the output switch contacts will close. These contacts have a maximum resistance of 2.5   and can conduct up to 250 mA. The maximum voltage that can be switched is 30VAC/60VDC. The output is a solid state switch closure and can conduct AC or DC.

MAXIMIZING PERFORMANCE

Extension Cables

Sensors which are calibrated with a probe extension cable must be operated with the extension cable to meet specifications. Operating without the extension cable will result in inaccurate measurements.

Probe Mounting

If multiple probes are mounted together, they must be separated by at least three probe diameters to prevent interference between the channels. The area within 3 probe diameters to the sides and 1.5 diameters behind should be kept clear of any metallic objects other than the object to be measured. Otherwise, custom calibration will be required.
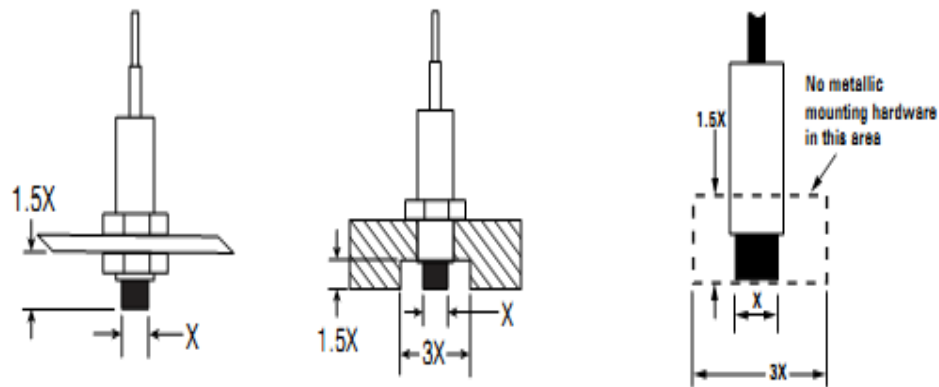
Ungrounded Targets

Ungrounded targets have the potential to inject noise into the sensor. If the output is unusually noisy, be sure the target is grounded. On moving/rotating targets this can be accomplished with a small metal brush or thin piece of metal which is connected to ground.

CONNECTING TO THE ECL202

Top Front **Top Rear**

- Remote Offset •
- Remote Offset +
- No Connection
- No Connection

- Remote Threshold -•
- Remote Threshold +
- Switch
- Switch

Eddy-Current Displacement Sensor **LION PRECISION**
All connections SELV (Safety Extra Low Voltage)

- Output Signal
- Output Signal Ground
- Power Ground
- 15-24V=== 170mA Input

- Bandwidth Select Jumper
- 100Hz
- 1kHz   (None=15kHz)
- 10kHz

Bottom Front **Bottom Rear**

ECL20X
FAR
NEAR
OFFSET
PROBE
LION PRECISION

SPECIFICATIONS:

Parameter		Specification	Notes
Power Requirement		15-24 VDC, 2.5 W	
Resolution @15kHz (Typical)[1, 2]	Nonferrous	0.006 to 0.008%F.S. (ECL202)	See calibration sheet for specifics
	Ferrous	0.3 µm or higher (ECL202e) 0.007 to 0.1%F.S. (ECL202) 0.3 µm or higher (ECL202e)	
Linearity		±0.2%F.S.	
Error Band1		±0.4%F.S.	
Analog Output1		0-10 VDC, 0 , 15 mA max	
Analog Output Update Rate		15 µS	
Setpoint Switch Output		Solid state switch closure: On state: 2.5 , 250 mA max Off state: 30 VAC/60 VDC max	
Remote Setpoint and Offset Inputs		15-24 VDC to activate, 3-7 mA	Optoisolator inputs
Driver Operating Environment		4°C-50°C, IP40	
Probe Operating Environment temp Temp	Standard Probes	-25°C to +125°C, IP67	
	High Probes	-25°C to +200°C, IP63	

Actual values depend on probe and range and are listed on the calibration certificate shipped with the product. Contact Lion Precision for replacement certificates.

In High EMI environments (10 V/m), output noise levels may rise to 30 mV causing resolution to be reduced to 0.3%.

EXPERIMENTAL RIG

1. Two metal plates were taken.

2. In each plate a hole (Radius: 59.5 mm) was made to accommodate the shell into it.

3. The shell was entered into the holes of the plates by press fit. The joints at the both ends of the shell with the plates were made in such a way that there was no gap between the shell and the holes of the plates.

4. A concrete base was made in order to make the set up more stable.

Figure: Simply supported cylindrical shell

PROCEDURE OF THE EXPERIMENT

The proximity sensor was positioned approximately in the range of 4.0 mm from the outer surface of the shell. The output of the sensor was connected to the oscilloscope.

A piece of Aluminum foil was stick with the outer surface of the shell by glue. Point to be added that this proximity sensor works on the metal surface too.
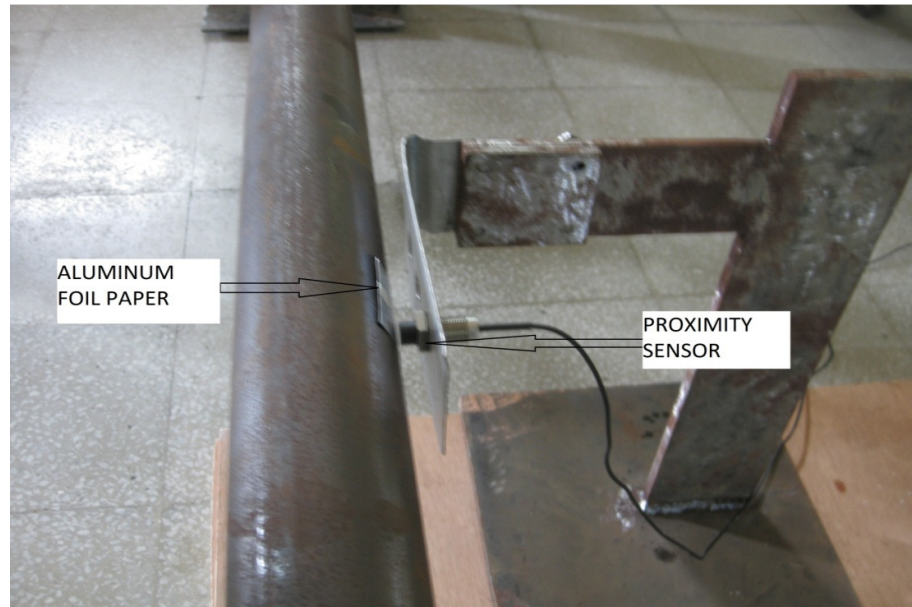

Figure: sensor and shell

The distance between the shell and sensor was determined by the green indicator of the probe.

The shell was hit by the hammer.

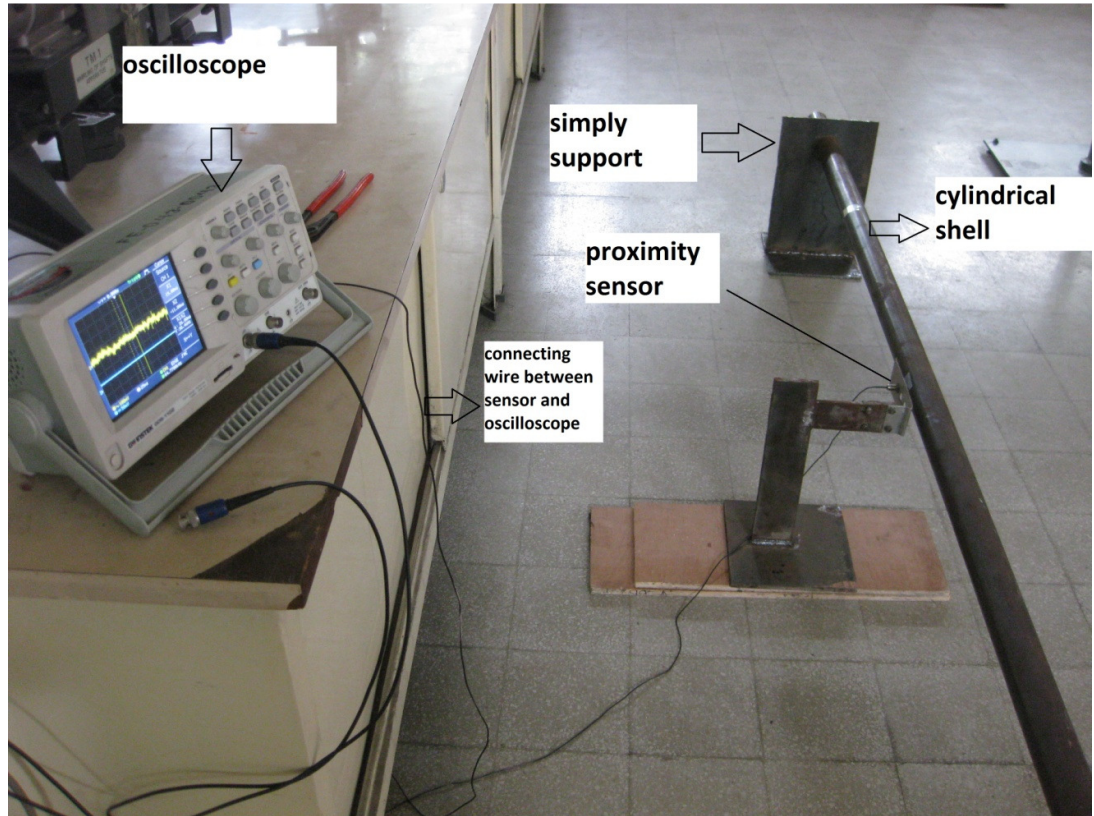The frequency was measured from the oscilloscope for the 1st mode.

Figure: Experimental set-up.

EXPERIMENTAL RESULT:

NATURAL FREQUENCY FOR THE FIRST MODE= **35 Hz**

CHAPTER 6

COMPARISON AND CONCLUSION

The comparison of the first natural frequency obtained by the three types of analyses are shown in the Table below

Type of Analysis	Frequency (Hz)	Difference with respect analytical analysis (%)
Numerical	34.07	4.27
Analytical	35.59	0.00
Experimental	35.00	1.66

From the above comparison, it can be seen that there is no significant difference between the analytical and experimental results, but there is a slight difference (4.27 %) in between the Numerical and Analytical results. To find out the reason of this difference, we have investigated the natural frequencies of the circular cylindrical shell of varying diameter, keeping the length and thickness constant and the results are shown below.

Outer Diameter (mm)	First Natural Frequency (Hz)		
	Numerical	Analytical	Difference with respect analytical analysis (%)
59.5	34.07	35.59	4.27
119	68.83	68.41	0.61
200	114.40	114.41	0.00009
300	82.14 or 82.26	82.25	0.0013 or 0.00012
400	80.71 or 80.75	80.75	0.0005 or 0.00

Analyzing the above results we can infer that the numerical and analytical results get closer as the radius of curvature of the shell increase. The reason may be for this is that the element we have used for 2D meshing namely "Plane 42" is very stiff for bending. Again in experimental analysis certainly little air damping was present which we have not considered during Numerical and Analytical analysis.

In the forced vibration analysis, we could not compare the Numerical and Analytical data because the damping effects were not properly taken into account which is beyond the scope of the current study. What we did was the consideration of air damping for analytical purpose and as there was no external

damping specified during numerical analysis therefore only the structural damping came into account by default.

FUTURE WORKS

In the present study we could not experimentally analyze forced vibration, so we are going to do that and also higher natural frequencies can be obtained experimentally by the process. We will also work on determination of structural damping theoretically so that results of analytical and numerical analysis for forced vibration can be compared. Then we are going to use dampers in all of the analyses processes to damp the natural frequency and reduce the response of the structure at these critical frequencies and examine the structure for different end conditions. Moreover we are going to analyze fluid induced vibration for both flow through and over the shell and try to introduce control mechanisms.

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