ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

MID SEMESTER EXAMINATION **SUMMER SEMESTER, 2021-2022**

DURATION: 1 HOUR 30 MINUTES **FULL MARKS: 75**

Math 4441: Probability and Statistics

Programmable calculators are not allowed. Do not write anything on the question paper.

Answer **all 3 (three)** questions. Figures in the right margin indicate full marks of questions whereas corresponding CO and PO are written within parentheses.

1. Consider a centralized wireless network where collision of packets are avoided by TDMA (Time-Division Multiple Access) scheduling. The success (or failure) of a packet delivery depends only on the wireless channel condition. If the wireless channel quality is good, the probability that a transmitted packet will be successfully delivered is 0.8. In contrast, if the wireless channel quality is poor, the probability that a transmitted packet will be successfully delivered is 0.6. Assume that the channel quality remains unchanged for a certain amount of time and after that it changes from the good state to the poor state and vice versa. Suppose the probability that the channel quality is good at a random time is 0.75 and the probability that the channel quality is poor is 0.25. Assume further that the probability of success (or failure) of a transmitted packet, at a certain channel quality (good or poor), is independent of the other packets transmitted at the same time instant (i.e., at the same channel quality).

   a) Suppose the delivery status of five randomly selected packets is inspected at a certain time. Assume the channel quality is same during this time, i.e., the channel quality is either good or poor, but it does not change during this time. 10 + 7 (CO1) (PO1)

      i. Find the probability that four of the five transmitted packets are delivered successfully.

      ii. Find the posterior probability that the channel condition was good at that time, given that four of the five transmitted packets were delivered successfully.

   b) Suppose that the delivery status of one additional packet, which was delivered at the same time, is inspected again. 8 + 5 (CO1) (PO1)

      i. Find the probability that the additional inspected packet is delivered successfully using the conditional law of total probability.

      ii. Find the new posterior probability that the channel condition was good at that time, given that four of the first five and the additional inspected packets were delivered successfully using the conditional Bayes theorem.

2. A computer executes two types of tasks, priority and non-priority, and operates in discrete time units (slots). A priority task arises with probability p at the beginning of each slot, independently of other slots, and requires one full slot to complete. A non-priority task is executed at a given slot only if no priority task is available. In this context, it may be important to know the probabilistic properties of the time intervals available for non-priority tasks.

   With this in mind, let us call a slot busy if within this slot, the computer executes a priority task, and otherwise, let us call it idle. We call a sequence of idle (or busy) slots, surrounded by busy (or idle, respectively) slots, an idle period (or busy period, respectively).

   a) Let T denote the time index of the first idle slot. Write down the PMF of T and justify it in no more than two sentences. 6 (CO1) (PO1)

   b) Suppose a non-priority task requires 5 slots to complete it. Assume no other non-priority task is present, and if any new non-priority task arrives, it will wait until this task is finished. Let S denote the number of slots required to complete the task. Write down the PMF of S and justify it in no more than two sentences. 7 (CO2) (PO2)

c) Find the probability that there will be 10 busy slots before 5 idle slots. 9 (CO2) (PO2)

3. Suppose that a professor will take a lab viva of five students one after another. The number of minutes required by any particular student to complete the viva has the exponential distribution for which the mean is 30. Suppose that the examination begins at 9:00 a.m.

   a) Determine the probability that the first student will complete the examination before 9:40 a.m. 7 (CO1) (PO1)

   b) Determine the probability that it will be at least 11:00 a.m. to complete the viva of the first three students. 10 (CO2) (PO2)

   c) Find the expected time required to complete the viva of all five students. 6 (CO1) (PO1)

Necessary Formulas

Law of Multiplication	$P[AB] = P[A\|B]P[B] = P[B\|A]P[A]$
Law of Total Probability	$P[A] = \sum_{i=1}^{n} P[A\|B_i]P[B_i]$
Bayes' Theorem	$P[B_i\|A] = \frac{P[A\|B_i]P[B_i]}{\sum_{i=1}^{n} P[A\|B_i]P[B_i]}$
Expectation of X	$E[X] = \sum_{x\in S_X} x\, P_X(x) = \int_{-\infty}^{+\infty} x\, f_X(x)dx$
Variance of X	$V[X] = E[X^2] - (E[X])^2$

PMF/PDF, expected value and variance of some known Random Variables

Distribution	PMF/PDF	Expected value	Variance
Bernoulli	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & otherwise \end{cases}$	$E[X] = p$	$Var[X] = p(1-p)$
Geometric	$P_X(x) = \begin{cases} p(1-p)^{x-1} & x \geq 1 \\ 0 & otherwise \end{cases}$	$E[X] = 1/p$	$Var[X] = (1-p)/p^2$
Binomial	$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 1, \ldots, n \\ 0 & otherwise \end{cases}$	$E[X] = np$	$Var[X] = np(1-p)$
Pascal	$P_X(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x = k, k+1, \ldots \\ 0 & otherwise \end{cases}$	$E[X] = k/p$	$Var[X] = k(1-p)/p^2$
Poisson	$P_X(x) = \begin{cases} \frac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = \alpha$ $\alpha = \lambda T$	$Var[X] = \alpha$
Hyper Geometric	$P_X(x) = \frac{\binom{r}{x}\binom{g}{n-x}}{\binom{r+g}{n}}$	$E[X] = \frac{rn}{r+g}$	
Uniform (discrete)	$P_X(x) = \begin{cases} \frac{1}{b-a+1}, & x = a, a+1, a+2, \ldots, b \\ 0, & \text{otherwise} \end{cases}$	$E[X] = \frac{a+b}{2}$	$Var[X] = \frac{(b-a)(b-a+2)}{12}$
Exponential	$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = \frac{1}{\lambda}$	$Var[X] = 1/\lambda^2$
Gaussian	$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} & \sigma > 0 \\ 0 & otherwise \end{cases}$	$E[X] = \mu$	$Var[X] = \sigma^2$
Uniform (Continuous)	$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & otherwise \end{cases}$	$E[X] = \frac{a+b}{2}$	$Var[X] = \frac{(b-a)^2}{12}$
Gamma	$f_X(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{\Gamma(r)}, & x \geq 0 \\ 0, & otherwise \end{cases}$	$E[X] = \frac{r}{\lambda}$	$V[X] = \frac{r}{\lambda^2}$