ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

SEMESTER FINAL EXAMINATION | **SUMMER SEMESTER, 2021-2022**

DURATION: 3 HOURS | **FULL MARKS: 150**

Math 4441: Probability and Statistics

Programmable calculators are not allowed. Do not write anything on the question paper.

Answer **all 6 (six)** questions. Figures in the right margin indicate full marks of questions whereas corresponding CO and PO are written within parentheses.

---

1. Suppose you roll a six-sided fair die twice and check the obtained numbers.

   a) Find the probability that the summation of the obtained numbers is even. — 8 (CO2) (PO1)

   b) Find the probability that both the obtained numbers are even when the summation of the obtained number is even. — 5 (CO2) (PO1)

   c) Assume that two different numbers have been rolled. Let X be the smaller number and Y be the larger number. Find: — 6×2 (CO3) (PO1)

      i. Joint probability mass function $f(x, y)$

      ii. $P(Y = 2X)$

2. An analysis of passenger data for a particular flight shows a 10% no-show rate. The capacity of that flight is 40. However, to maximize revenue, the airline has decided to sell more tickets than the actual capacity of the flight.

   a) If the airline sells 44 tickets, find the probability that everyone who shows up can board the flight? — 9 (CO2) (PO1)

   b) Determine the probability that at least 6 tickets need to be sold in order to get a no-show passenger. — 8 (CO2) (PO1)

   c) Assuming the flight's capacity consists of 6 first-class and the remaining 34 world-traveler seats and the airline sells a maximum of 8 first-class tickets, determine the expected number of first-class passengers that will show up. — 8 (CO3) (PO2)

3. Suppose we want to study how often lightning strikes occur at a particular location. Historical data suggests lightning strikes occur on average once every 30 minutes at this location.

   a) Find the probability of 2 lightning strikes occuring in one hour. — 7 (CO2) (PO1)

   b) If a lightning strike just takes place, find the probability that it will take more than 40 minutes for the next strike. — 8 (CO2) (PO1)

   c) Determine the probability that the 4th lightning strikes within 100 to 140 minutes. — 10 (CO2) (PO1)

4. Let X and Y be jointly continuous random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} cx+1 & x,y \geq 0, x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

Now find the following:

a) Value of the constant c.

b) Marginal Probability Density function for X and Y.

c) Expected value of X and Y.

d) Covariance of X and Y.

e) Correlation Coefficient between X and Y.

5. A teacher picked 6 students randomly in her class to analyze their scores in exams before and after taking extra tutorial classes. Table 1 lists the score of the students before and after the tutorial classes. According to her claim, the average score before the tutorial was not greater than 70 with a standard deviation of 12 and this average score changed after the tutorial classes. Let the confidence level be 95% and $\alpha = 0.05$

Table 1: Scores of Students for Question 5.

	S_1	S_2	S_3	S_4	S_5	S_6
Pre-Score	78	67	56	78	96	82
Post-Score	80	69	70	79	96	84

a) Find the critical value and determine whether the teacher's claim about the average score before the tutorial was correct or not.

b) Assume that the true population is 72. Find the power of the test of Question 5.a).

c) Test the effectiveness of the tutorial classes from the given data.

6. An airport official wants to assess if the flights from one airline (Airline 1) are less delayed but much variate than flights from another airline (Airline 2). A random sample of 10 flights for Airline 1 shows an average of 9.5 minutes delay with a standard deviation of 6 minutes. A random sample of 10 flights for Airline 2 shows an average of 12.63 minutes delay with a standard deviation of 3 minutes. Assume delay times are normally distributed.

a) Test the equality of the variance of delay for airlines with 5% significance.

b) Using the assumption and the similar level of significance as 6.a), does the test statistic support that the flights from Airline 1 are less delayed than flights from Airline 2?

c) Find the P-value for the difference of mean delays of the airlines and determine whether the official is correct or not.

Formulas of Probability

Expected Value, $\mathrm{E}[X] = \sum_{i=1}^{n} (x_i) P(x_i)$	Variance, $\mathrm{Var}(X) = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2$

Geometric Distribution

$$\text{PMF, } P(x) = \begin{cases} (1-p)^{(x-1)}p & x \geq 1 \\ 0 & \text{Otherwise} \end{cases}$$

Expected Value, $\mathrm{E}[X] = \frac{1}{p}$

Variance, $\mathrm{Var}[X] = \frac{1-p}{p^2}$

Binomial Distribution

$$\text{PMF, } P(x) = \begin{cases} \binom{n}{x} (p)^x (1-p)^{(n-x)} & x = 0, 1, 2, \cdots, n \\ 0 & \text{Otherwise} \end{cases}$$

Expected Value, $\mathrm{E}[X] = np$

Variance, $\mathrm{Var}[X] = np(1-p)$

Negative Binomial Distribution

$$\text{PMF, } P(x) = \begin{cases} \binom{x-1}{k-1} (p)^k (1-p)^{(x-k)} & x \geq k \\ 0 & \text{Otherwise} \end{cases}$$

Expected Value, $\mathrm{E}[X] = k\frac{1-p}{p}$

Variance, $\mathrm{Var}[X] = k\frac{1-p}{p^2}$

HyperGeometric Distribution

$$\text{PMF, } P(x) = \begin{cases} \dfrac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}} & a \geq x; \quad b \geq n-x \\ 0 & \text{Otherwise} \end{cases}$$

Expected Value, $\mathrm{E}[X] = \frac{na}{a+b}$

Variance, $\mathrm{Var}[X] = \frac{nab(a+b-n)}{(a+b)^2(a+b-1)}$

Truncated Geometric Distribution

$$\text{PMF, } P(x) = \begin{cases} (1-p)^{(x-1)}p & 1 \leq x \leq R-1 \\ (1-p)^{(R-1)} & x = R \\ 0 & \text{Otherwise} \end{cases}$$

Discrete Uniform Distribution

$$\text{PMF, } P(x) = \begin{cases} \frac{1}{n} & x \in S_X \\ 0 & \text{Otherwise} \end{cases}$$

Expected Value, $\mathrm{E}[X] = \frac{n+1}{2}$

Variance, $\mathrm{Var}[X] = \frac{n^2-1}{12}$

Poisson Distribution

$$\text{PMF, } f(x) = \begin{cases} \frac{e^{-\alpha}\alpha^x}{x!} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Expected Value, $\mathrm{E}[X] = \alpha$

Variance, $\mathrm{Var}[X] = \alpha$

Continuous Uniform Distribution

$$\text{PDF, } f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}$$

$$\text{CDF, } F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$$

Expected Value, $\mathrm{E}[X] = \frac{a+b}{2}$

Variance, $\mathrm{Var}[X] = \frac{(b-a)^2}{12}$

Continued

Exponential Distribution		
PDF, $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$	CDF, $F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	Expected Value, $\mathrm{E}[X] = \frac{1}{\lambda}$
		Variance, $\mathrm{Var}[X] = \frac{1}{\lambda^2}$

Gamma Distribution		
PDF, $f(x) = \begin{cases} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} & x \geq k \\ 0 & \text{Otherwise} \end{cases}$	CDF, $F(x) = \begin{cases} 1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-\lambda x} (\lambda x)^n & x \geq 0 \\ 0 & x < 0 \end{cases}$	Expected Value, $\mathrm{E}[X] = \frac{1}{\lambda}(k-1)$
		Variance, $\mathrm{Var}[X] = \frac{k}{\lambda^2}$

Discrete Joint Probability Distribution	
CDF, $F(x) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f_{XY}(x,y)dxdy$	Marginal PDFs, $f_X(x) = \sum_{-\infty}^{\infty} f_{XY}(x,y)dy$, for all x $f_Y(y) = \sum_{-\infty}^{\infty} f_{XY}(x,y)dx$, for all y

Continuous Joint Probability Distribution	
CDF, $F(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)dxdy$	Marginal PDFs, $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$, for all x $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$, for all y

Covariance, $Cov(X,Y) = E[XY] - (EX)(EY)$	Correlation Coefficient, $\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$

Formulas of Statistics

Equality of Two Means			
Assumption	Test Statistic TS	Significance Level α Test	p-Value if $TS = t$
σ_1, σ_2 known	$\frac{\bar{X}-\bar{Y}}{\sqrt{\sigma_1^2/n+\sigma_2^2/m}}$	Reject if $\lvert TS\rvert > z_{\alpha/2}$	$2P\{Z \geq \lvert t\rvert\}$
$\sigma_1 = \sigma_2$	$\frac{\bar{X}-\bar{Y}}{\sqrt{\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}\sqrt{1/n+1/m}}$	Reject if $\lvert TS\rvert > t_{\alpha/2,n+m-2}$	$2P\{T_{n+m-2} \geq \lvert t\rvert\}$
n, m large	$\frac{\bar{X}-\bar{Y}}{\sqrt{S_1^2/n+S_2^2/m}}$	Reject if $\lvert TS\rvert > z_{\alpha/2}$	$2P\{Z \geq \lvert t\rvert\}$
σ_1, σ_2 unknown	$\frac{\bar{X}-\bar{Y}}{\sqrt{S_1^2/n+S_2^2/m}}$	Reject if $\lvert TS\rvert > t_{\alpha/2}, \min(n-1, m-1)$	$2P\{T_{\min(n-1,m-1)} \geq \lvert t\rvert\}$

Chi-Square Distribution	F Distribution
$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$	$\mathrm{F} = \frac{\left[\frac{(s1)^2}{(\sigma 1)^2}\right]}{\left[\frac{(s2)^2}{(\sigma 2)^2}\right]} = \frac{(s_1)^2}{(s_2)^2}$

Alpha Level Critical Value (Z-score)

α	$\alpha/2$	$Z_{\alpha/2}$
0.1	0.05	1.645
0.05	0.025	1.96
0.025	0.0125	2.241
0.01	0.005	2.576
0.005	0.0025	2.807

Z Distribution

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

Student's T Distribution

ν \ α	.40	.25	.10	.05	.025	.01	.005	.0025	.001
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31
2	.289	.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326
3	.277	.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213
4	.271	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173
5	.267	.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893
6	.265	.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208
7	.263	.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785
8	.262	.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501
9	.261	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297
10	.260	.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144
11	.260	.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025
12	.259	.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930
13	.259	.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852
14	.258	.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787
15	.258	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733
16	.258	.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450

F Distribution for $\alpha = 0.05$

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92

Degrees of freedom for the denominator (ν_2)

Degrees of freedom for the numerator (ν_1)