ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

MID SEMESTER EXAMINATION **WINTER SEMESTER, 2022-2023**

DURATION: 1 HOUR 30 MINUTES **FULL MARKS: 75**

CSE 6261: Advanced Probability and Stochastic Process

Programmable calculators are not allowed. Do not write anything on the question paper.

Answer **all 6 (six)** questions. Figures in the right margin indicate full marks of questions.

1. The Navy's new underwater digital communication system is not perfect. In any sufficiently long period of operation, the number of communication errors can be modeled as a Poisson random variable X, i.e., $X \sim Poisson(\alpha)$, where the parameter α represents the average number of errors occurring in the time period. 15

   The contractor, Poseidon Systems Incorporated, responsible for developing this system, having realized that the communication system is not perfect, has implemented an error detection system at the receiver. Unfortunately, the error detection system is not perfect either. Given that X errors occur, only Y of those errors are detected, $Y \leq X$. Y is a random variable with a binomial distribution conditioned on random variable X, i.e., $Y|X \sim binom(x, p)$, where p is the probability of detecting a single error.

   To make matters worse, the contractor, Poseidon Systems Incorporated, who had never heard of the Poisson distribution, thought that their system was being dubbed by the Navy as a "Poseidon random variable," and threatened to sue.

   The admiral in charge of development has called for a performance analysis of the overall system and has required the contractor to compute $f_Y(y)$, the marginal PMF for the number of errors detected, and the conditional PMF $f_{X|Y}(x|y)$ for the number of errors occurring given the number of errors corrected. The admiral is a graduate of the Naval Postgraduate School with a master's degree in operations research, and claims that both of these distributions are Poisson.

   Find the marginal PDF of Y and the conditional PMF of $X|Y$ to show whether the admiral is correct or not.

2. A Canadian pharmaceutical company has developed three types of Corona virus vaccines. Suppose that vaccines I, II, and III are effective 92%, 88%, and 96% of the time, respectively. Assume that a person reacts to the Corona vaccine independently of other people. The company sends to Sangala, a fictitious nation, hundreds of qualified vaccine carrier containers containing 100 ml vaccine bottles each filled with one of the three types of the Corona vaccines. Suppose that the bottles of 40% of the containers are filled with only the type I vaccine, and the bottles of 35% and 25% of the containers are filled only with type II and type III vaccines, respectively. 15

   In Sangala, a physician assistant takes one of the containers at random to a small village and using its contents, vaccinates its entire 200-people population. It is certain that sooner or later every member of the village will be exposed to the Corona virus. Find the probability that at most 6 of the villagers will become sick with Corona.

3. A point is selected at random from the disk 10

   $$R = \{(x, y) \in R^2 \, : \, x^2 + y^2 \leq 1\}.$$

   Let X be the x-coordinate and Y be the y-coordinate of the point selected. Find the joint PDF and marginal PDFs of X and Y. Determine if X and Y are independent random variables.

4. The joint probability density function (JPDF) of random variables X and Y be given by

$$f_{X,Y}(x,y) = \begin{cases} 1, & \text{for } |y| < x, 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Find $E[X|Y=y]$ and $E[Y|X=x]$.

5. A straight rod is formed by connecting three sections A, B, and C, each of which is manufactured on a different machine. The length of section A, in inches, has the normal distribution with mean 20 and variance 0.04. The length of section B, in inches, has the normal distribution with mean 14 and variance 0.01. The length of section C, in inches, has the normal distribution with mean 26 and variance 0.04. As indicated in Fig. 1, the three sections are joined so that there is an overlap of 2 inches at each connection. Suppose that the rod can be used in the construction of an airplane wing if its total length in inches is between 55.7 and 56.3. Find the probability that the rod can be used in the construction of the airplane wing. 10

[Note: The sum of a number of Gaussian random variables is also a Gaussian random variable.]

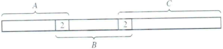

Figure 1: Figure for Question 5

6. Customers enter a department store following the Poisson distribution at the rate of three per minute. If 30% of them buy nothing, 20% pay cash, 40% use credit cards, and 10% use bKash, find the probability that in five operating minutes of the store, five customers use credit cards, two use bKash, and three pay cash. 15

Appendix

Necessary Equations/formulas

$$P_{X|A}(x) = \frac{P_X(x)}{P[A]}, x \in A$$

$$P_{XY}(x, y) = P_{X|Y}(x|y)P_Y(y)$$

$$P_{XY}(x, y) = P_X(x)P_Y(y), if\ X \perp Y$$

$$P_X(x) = \sum_y P_{XY}(x, y) = \sum_y P_{X|Y}(x|y)P_Y(y)$$

$$E[X] = \sum_y E[X|Y = y]P_Y(y)$$

$$H[X] = -\sum_x P_X(x) \log P_X(x)$$

$$H[X, Y] = -\sum_y \sum_x P_{XY}(x, y) \log P_{XY}(x, y)$$

$$H[X, Y] = H[X] + HY|X]$$

$$I[X; Y] = H[X] - H[X|Y]$$

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} P_X(x) = \int_{-\infty}^{+\infty} e^{tx} f_X(x) dx$$

$$G_X(t) = E[z^X] = \sum_x z^x P_X(x)$$

Necessary Formulas

Law of Multiplication	$P[AB] = P[A\|B]P[B] = P[B\|A]P[A]$
Law of Total Probability	$P[A] = \sum_{i=1}^{n} P[A\|B_i]P[B_i]$
Bayes' Theorem	$P[B_i\|A] = \frac{P[A\|B_i]P[B_i]}{\sum_{i=1}^{n} P[A\|B_i]P[B_i]}$
Expectation of X	$E[X] = \sum_{x \in S_X} x P_X(x) = \int_{-\infty}^{+\infty} x f_X(x) dx$
Variance of X	$V[X] = E[X^2] - (E[X])^2$

PMF/PDF, expected value and variance of some known Random Variables

Distribution	PMF/PDF	Expected value	Variance
Bernoulli	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & otherwise \end{cases}$	$E[X] = p$	$Var[X] = p(1-p)$
Geometric	$P_X(x) = \begin{cases} p(1-p)^{x-1} & x \geq 1 \\ 0 & otherwise \end{cases}$	$E[X] = 1/p$	$Var[X] = (1-p)/p^2$
Binomial	$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 1, \ldots, n \\ 0 & otherwise \end{cases}$	$E[X] = np$	$Var[X] = np(1-p)$
Pascal	$P_X(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x = k, k+1, \ldots \\ 0 & otherwise \end{cases}$	$E[X] = k/p$	$Var[X] = k(1-p)/p^2$
Poisson	$P_X(x) = \begin{cases} \dfrac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = \alpha$ $\alpha = \lambda T$	$Var[X] = \alpha$
Hyper Geometric	$P_X(x) = \dfrac{\binom{r}{x}\binom{g}{n-x}}{\binom{r+g}{n}}$	$E[X] = \dfrac{rn}{r+g}$	
Uniform (discrete)	$P_X(x) = \begin{cases} \dfrac{1}{b-a+1}, & x = a, a+1, a+2, \ldots, b \\ 0, & \text{otherwise} \end{cases}$	$E[X] = \dfrac{a+b}{2}$	$Var[X] = \dfrac{(b-a)(b-a+2)}{12}$
Exponential	$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = \dfrac{1}{\lambda}$	$Var[X] = 1/\lambda^2$
Gaussian	$f_X(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} & \sigma > 0 \\ 0 & otherwise \end{cases}$	$E[X] = \mu$	$Var[X] = \sigma^2$
Uniform (Continuous)	$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x < b \\ 0, & otherwise \end{cases}$	$E[X] = \dfrac{a+b}{2}$	$Var[X] = \dfrac{(b-a)^2}{12}$
Gamma	$f_X(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma(r)}, & x \geq 0 \\ 0, & otherwise \end{cases}$	$E[X] = \dfrac{r}{\lambda}$	$V[X] = \dfrac{r}{\lambda^2}$
Multivariate Hyper geometric	$P_{X_1,\ldots,X_r}(x_1, \ldots, x_r) = \dfrac{\binom{n_1}{x_1}\binom{n_2}{x_2}\ldots\binom{n_r}{x_r}}{\binom{n_1+\ldots+n_r}{x_1+\cdots+x_r}}$		
Multinomial	$P_{X_1,\ldots,X_r}(x_1, \ldots, x_r) = \binom{n}{x_1 \quad \ldots \quad x_r} p_1^{x_1} \ldots p_r^{x_r}$		

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

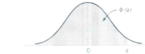

TABLE III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967