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ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

SEMESTER FINAL EXAMINATION WINTER SEMESTER, 2022-2023

DURATION: 3 HOURS FULL MARKS: 150

Math 4341: Linear Algebra

Programmable calculators are not allowed. Do not write anything on the question paper.

Answer **all 6 (six)** questions. Figures in the right margin indicate full marks of questions whereas corresponding CO and PO are written within parentheses.

1. a) Let $a, b \in \mathbb{R}$, and let 10+5 (CO1) (PO1)

$$A = \begin{bmatrix} 1 & 2 & 3 & a \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & b \end{bmatrix}$$

   i. What are the dimensions of the four subspaces associated with the matrix A? These values will depend on the values of a and b, and you should distinguish all cases.
   ii. For $a = b = 1$, give a basis for the columnspace of A. Is this also a basis for $\mathbb{R}^3$? Justify your answer.

   b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation satisfying: 10 (CO1) (PO1)

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -10 \\ 8 \end{bmatrix}.$$

   Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

2. a) If you convert matrix A (shown below) to row-reduced echelon form by the usual row elimination steps, you will achieve $R = rref(A)$ as: 20 (CO3) (PO1)

$$A = \begin{bmatrix} 1 & 2 & 1 & -7 \\ 2 & 4 & 1 & -5 \\ 1 & 2 & 2 & -16 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   i. What are the minimum and the maximum number of columns of A that form a dependent and independent set of vectors, respectively?
   ii. Give an orthonormal basis for the rowspace of A. Note that, it is not $C(A)$.
   iii. Given the vector $b = (2\ 5\ -9\ 3)^T$, compute a vector p that is closest to b in the rowspace $C(A^T)$.
   iv. In terms of your calculated p in Question 2.a)iii., what is the closest vector to b in the nullspace $N(A)$?

   b) What is the value of the following determinant for any values of a, b and c? 5 (CO1) (PO1)

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$$

3. a) Imagine a plane $z = Cx + Dy + E$ that is closest (in the least squares sense) to these four measurements: 15 (CO2) (PO1)

$$\begin{gathered}\text{At } x = 1; y = 0 \text{ measurement gives } z = 1\\ \text{At } x = 1; y = 2 \text{ measurement gives } z = 3\\ \text{At } x = 0; y = 1 \text{ measurement gives } z = 5\\ \text{At } x = 0; y = 2 \text{ measurement gives } z = 0\end{gathered}$$

Show that this system $Ax = b$ has no solution. Find the best least squares solution $\widehat{z} = (\widehat{C}, \widehat{D}, \widehat{E})$.

b) A is an $m \times n$ matrix. Suppose $Ax = b$ has at least one solution for every b. 10 (CO1) (PO1)

i. What is the rank of A?

ii. Describe all vectors in the nullspace of A^T.

iii. Determine whether the equation $A^T y = c$ has (0 or 1), (1 or ∞), (0 or ∞), or (1) solution for every c.

4. a) Suppose matrix A has eigenvalues $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$ with corresponding eigenvectors: 10 (CO2) (PO1)

$$x_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad x_2 = \begin{bmatrix}0\\1\\1\end{bmatrix} \quad x_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Find the matrix A.

b) Let V be a two-dimensional subspace in $\mathbb{R}^3$ consisting of vectors $[x \;\; y \;\; z]^T$ satisfying: 15 (CO1) (PO1)

$$x + 2y - 5z = 0$$

i. Find a 3×2 matrix A whose columnspace is V.

ii. Find an orthonormal basis for V.

5. a) Consider a sequence of numbers defined using the following terms: 5 + 10+5 (CO3) (PO1)

$$\begin{gathered}G_0 = 0\\ G_1 = 1/2\\ G_{k+2} = (G_{k+1} + G_k)/2\end{gathered}$$

i. Set up a 2×2 matrix A to get from $\begin{bmatrix}G_{k+1}\\G_k\end{bmatrix}$ to $\begin{bmatrix}G_{k+2}\\G_{k+1}\end{bmatrix}$.

ii. Find an explicit formula for G_k.

iii. What is G_{1000}? What is the limit of G_k as $k \to \infty$?

b) Suppose a 3×3 matrix A has $row_1 + row_2 = row_3$. Explain why $Ax = [1 \;\; 0 \;\; 0]^T$ can not have a solution. 5 (CO1) (PO1)

6. a) Compute the Singular Value Decomposition $A = U\Sigma V^T$ for $A = \begin{bmatrix}0 & 1\\-1 & 1\\1 & 0\end{bmatrix}$. 15 (CO3) (PO1)

b) The matrix A has a varying $1 - x$ in the (1, 2) position: 10 (CO1) (PO1)

$$A = \begin{bmatrix}2 & 1-x & 0 & 0\\1 & 1 & 1 & 1\\1 & 1 & 2 & 4\\1 & 1 & 3 & 9\end{bmatrix}$$

Considering the properties of the determinant, show that det(A) is a linear function of x. For any x, compute det(A). For which value of x is the matrix singular?