27

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SEMESTER FINAL EXAMINATION | WINTER SEMESTER, 2022-2023
DURATION: 3 HOURS | FULL MARKS: 150

Math 4543: Numerical Methods

Programmable calculators are not allowed. Do not write anything on the question paper.
Answer **all 6 (six)** questions. Figures in the right margin indicate full marks of questions whereas corresponding CO and PO are written within parentheses.

1. a) What are the steps of problem-solving in engineering? With a suitable example, briefly explain how numerical methods are used to solve an engineering problem. $2+1$ (CO2) (PO1)

   b) i. Suppose, R and k are arbitrary numbers such that $R > 0$ and $k \neq 0$. Prove that the Newton-Raphson formula for finding the kth root of R, which is mathematically denoted as $\sqrt[k]{R}$, is as shown in Equation-1. 5 (CO2) (PO1)

   $$x_{i+1} = \frac{1}{k}\left[(k-1)x_i + \frac{R}{x_i^{k-1}}\right] \qquad (1)$$

   ii. Draw a rough sketch of the graph of two functions $f_1(x)$ and $f_2(x)$ given that — $2+2$ (CO3) (PO2)
      - For $f_1(x)$, False Position method performs better than the Bisection method.
      - For $f_2(x)$, Bisection method performs better than the False Position method.

      Here, *"performs better"* means that it takes a lesser number of iterations to converge to the root. You have the liberty to choose the initial guesses $\langle x_l, x_u\rangle$ as well.

   iii. Derive the formula for the Secant method geometrically. 4 (CO4) (PO1)

   c) The *Bronze Ratio*, often denoted by the Greek letter β (beta), is the ratio of two quantities a and b, such that $a > b > 0$, if it satisfies the equality $\frac{3a+b}{a} = \frac{a}{b}$. The constant β is equal to the positive root of a particular function $f(x)$. The numerical value of that root is,

   $$x^{+}_{root} = \beta = \frac{3+\sqrt{13}}{2} = 3.302775638\ldots$$

   i. From the given scenario, formulate the function $f(x)$. 3 (CO1) (PO2)

   ii. Apply the Secant method to estimate the value of the bronze ratio β. The initial guesses are $x_{-1} = 1.5$ and $x_0 = 2$. Demonstrating the step-by-step mathematical procedure, conduct 4 iterations, and find the relative approximate error ($|\epsilon_a|\%$) and the number of significant digits that are at least correct (m) at the end of each iteration. Draw Table 1 on your answer script and fill it out after performing the necessary calculations. 11 (CO2) (PO1)

Table 1: The relevant values obtained in the answer of Question 1.c)

Iteration	x_{i-1}	x_i	x_{i+1}	$\lvert\epsilon_a\rvert\%$	m	$f(x_{i+1})$
1	1.5	2				
2						
3						
4						

2. a) "The magnitude of the true error and the magnitude of the approximate error do not necessarily reflect how bad the error is." — Do you agree? Justify your answer. 4 (CO3) (PO1)

b) The Taylor series for a function $f(x)$ is — 15 (CO2) (PO1)

$$f(x+h) = f(x) + f'(x)\frac{h}{1!} + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4!} + f'''''(x)\frac{h^5}{5!} + \dots$$

Using the Taylor series (with at least the first 5 terms), derive the Maclaurin series of

$$f(x) = \operatorname{cosec}(x)$$

c) We know, $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$. Using this information and the Taylor series (with at least the first 5 terms) approximate the value of $\sin(2)$. 8 (CO2) (PO1)

Note: The unit of angular measure that has been used here is radians (rad).

3. a) Prove that a polynomial of degree n or less that passes through $n+1$ data points is unique. 6 (CO4) (PO1)

b) Compare and contrast the following methods of interpolation — Direct method, Newton's Divided Difference method, and Lagrangian method. 4 (CO3) (PO1)

c) A thermistor, also known as thermal resistor, is a semiconductor type of resistor whose resistance is strongly dependent on temperature. To measure temperature using a thermistor, the manufacturers provide users with a Temperature (T) vs. Resistance (R) calibration curve. If we measure the resistance of the thermistor using an Ohmmeter, then we can refer to the aforementioned curve and determine the corresponding temperature value. Table 2 portrays multiple recorded observations involving a thermistor. 8 (CO2) (PO1)

Table 2: Temperature T of a thermistor as a function of its resistance R for Question-3.c)

Resistance, R (Ω)	**Temperature, T** ($^\circ C$)
1101	25.113
911.3	30.131
636	40.12
451.1	50.128

Determine the value of the temperature T when the thermistor resistance is $R = 754.8\Omega$ using the Newton's Divided Difference method of interpolation and a second order polynomial.

4. a) The general linear regression model for predicting the response for a given set of n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ is 6 + 6 (CO4) (PO1)

$$y = a_0 + a_1 x$$

where a_0 and a_1 are the two parameters of the regression model.

i. Derive the formulae for finding the optimal values of a_0 and a_1.

ii. Prove that the values of a_0 and a_1 obtained using the formulae derived in Question-4.a)(i) correspond to the absolute minimum of the optimization criterion used.

b) Suppose you are given a set of n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ and you want to regress the data to an mth order Polynomial model, where $m < n$. How would you choose a suitable value of m? 5 (CO3) (PO1)

c) Sodium borohydride ($NaBH_4$) is an inorganic compound that can be used as fuel for fuel cells. The Overpotential (η) vs Current (I) data depicted in Table 3 was obtained during a study conducted to assess its electrochemical kinetics. 8 (CO2) (PO1)

Table 3: Electrochemical Kinetics data of $NaBH_4$ for Question-4.c)

Overpotential, η (V)	**Current,** I (A)
−0.29563	0.00226
−0.24346	0.00212
−0.19012	0.00206
−0.18772	0.00202
−0.13407	0.00199
−0.0861	0.00195

The relationship that exists between the overpotential (η) and current (I) can be represented using the Logarithmic model (also known as the Tafel equation) —

$$\eta = a + b \ln I$$

where a is an electrochemical kinetics parameter of borohydride on the electrode and b is the experimental Tafel slope parameter. Determine the value of a and b using the data points provided in Table 3. You are allowed to apply data transformation/linearization.

5. a) i. Using the method of coefficients, derive the formula for the Single-segment Trapezoidal rule. 7 + 6 (CO3) (PO1)

ii. Applying the formula from Question-5.a)(i), derive the formula for the Multi-segment Trapezoidal rule.

b) The velocity v of a body is given by 5 + 4 (CO2) (PO1)

$$v(t) = \begin{cases} 2t; & \text{if } 1 \leq t \leq 5 \\ 5t^2 + 3; & \text{if } 5 < t \leq 14 \end{cases}$$

where the time t is given in seconds (s), and v is given in meters per second (m s^{-1}).

i. Using 2-segment Simpson's 1/3 rule, approximate the distance (S) in meters covered by the body from $t = 2$s to $t = 9$s.

ii. Analytically determine the exact value of the covered distance (S_{exact}) and use it to calculate the true error (E_t) and the absolute relative true error ($|\epsilon_t|\%$).

c) Define Eigenvector and Eigenvalue. 3 (CO3) (PO1)

6. a) Compare and contrast the following numerical algorithms for solving 1st-order ordinary differential equations — Euler's method, Runge-Kutta 2nd order method, and Runge-Kutta 4th order method. 4 (CO3) (PO1)

b) Determine the approximate value of 6 (CO2) (PO1)

$$I = \int_5^8 6x^3 dx$$

using Euler's method. Use a step size of $h = 1.5$.

c) i. Determine the *minimum* number of significant digits in each of these numerical values 3 + 2 (CO4) (PO1)

(a) 069420 (b) 6.94200×10^5 (c) 0.069420

(d) 69.00420 (e) 69.0 (f) 69420

ii. Determine the order and degree of each of these higher-order differential equations

(a) $x\dfrac{dy}{dx} - \left(\dfrac{d^2y}{dx^2}\right)^3 + 5 = 0$ (b) $\left(\dfrac{dy}{dx}\right)^2 - 4xy = 3$

(c) $\dfrac{d^3y}{dx^3} + x^4\dfrac{d^2y}{dx^2}\left(\dfrac{dy}{dx}\right)^4 = \sin x$ (d) $\dfrac{dy}{dx} + xy = x^2$

d) i. Determine the LU Decomposition of the 3×3 matrix $5+5$ (CO2) (PO1)

$$A = \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix}$$

ii. Suppose $u = u(x, y)$ is a function of two variables that we only know at discrete grid points (x_i, y_j) given in the matrix

$$[u_{i,j}] = \begin{bmatrix} 5.1 & 6.5 & 7.5 & 8.1 & 8.4 \\ 5.5 & 6.8 & 7.8 & 8.3 & 8.9 \\ 5.5 & 6.9 & 9.0 & 8.4 & 9.1 \\ 5.4 & 9.6 & 9.1 & 8.6 & 9.4 \end{bmatrix}$$

Find the approximate value of the following partial derivatives

(a) $u_x(x_2, y_4)$ (b) $u_y(x_2, y_4)$ (c) $u_{xx}(x_2, y_4)$

(d) $u_{yy}(x_2, y_4)$ (e) $u_{xy}(x_2, y_4)$

using the central difference formula. Consider $h = 0.5$ and $k = 0.2$.
Note: Assume 1-based indexing.