42

ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

SEMESTER FINAL EXAMINATION — **WINTER SEMESTER, 2022-2023**

DURATION: 3 HOURS — **FULL MARKS: 150**

Math 4741: Mathematical Analysis

Programmable calculators are not allowed. Do not write anything on the question paper.

Answer **all 6 (six)** questions. Figures in the right margin indicate full marks of questions whereas corresponding CO and PO are written within parentheses.

---

1. a) Suppose that customers arrive at a single-server service station in accordance with a Poisson process having rate λ and that the service time is exponential at a rate μ. Consider that there is a finite number of customers that could be in the system at the same time. Derive the probability that there are n customers in the system. 10 (CO2) (PO2)

   b) The railway marshalling yard is sufficient only for 9 trains (there being 10 lines, one of which is earmarked for the shunting engine to reverse itself from the crest of the hump to the rear of the train). Trains arrive at the rate of 30 trains per day, inter-arrival time follows an exponential distribution and service time distribution is also exponential with an average of 36 minutes. Calculate the following: 10 (CO2) (PO2)
      i. The probability that the yard is empty.
      ii. The average line length (average number of trains in the system).
      iii. The average amount of time a train will have to wait for service in the system.
      iv. The average number of trains in the queue.

   c) A telephone exchange has two long distance operators. It is observed that, during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length of 5 minutes. What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day? 5 (CO2) (PO2)

2. a) Beverly has a radio that works on a single battery. As soon as the battery in use fails, Beverly immediately replaces it with a new battery. If the lifetime of a battery (in hours) is distributed uniformly over the interval (30, 60), then at what rate does Beverly have to change batteries? Again, suppose that Beverly does not keep any surplus batteries on hand, and so each time a failure occurs she must go and buy a new battery. If the amount of time it takes for her to get a new battery is uniformly distributed over (0, 1), then what is the average rate that Beverly changes batteries? 5 (CO3) (PO2)

   b) The lifetime of a car is a continuous random variable having a distribution H and probability density h. Mr. Tutul has a policy that he buys a new car as soon as his old one either breaks down or reaches the age of T years. Suppose that a new car costs $C1$ dollars and also that an additional cost of $C2$ dollars is incurred whenever Mr. Tutul's car breaks down. Under the assumption that a used car has no resale value, what is Mr. Tutul's long-run average cost? Let the lifetime of a car (in years) is uniformly distributed over (0,10), and suppose that $C1$ is 3 (thousand) dollars and $C2$ is $\frac{1}{2}$ (thousand) dollars. What value of T minimizes Mr. Tutul's long-run average cost? 15 (CO4) (PO3)

   c) A truck driver regularly drives round trips from A to B and then back to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is uniformly distributed between 40 and 60; each time he drives from B to A, he drives at a fixed speed that is equally likely to be either 40 or 60. In the long run, what proportion of his driving time is spent going to B? 5 (CO4) (PO3)

3. a) Consider the mood of an individual (Mr. Minhaz) is considered as a three-state (cheerful (C), so-so (S), or glum (G)) Markov Chain having a transition probability matrix:

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

i. Suppose Mr. Minhaz is currently in a cheerful mood. What is the probability that he is not in a glum mood on any of the following three days? 8 (CO2) (PO2)

ii. In the long run, what proportion of time is the process in each of the three states? 7 (CO3) (PO2)

b) Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0? 10 (CO4) (PO3)

4. a) Consider the following transition probability matrix with states 0, 1 and 2: 10 (CO2) (PO2)

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 0.4 & 0.4 \\ 0 & 0.7 & 0.3 \end{bmatrix}$$

Determine the class of the Markov Chain whether they are recurrent or transient. Find the mean recurrence time for the recurrent states.

b) Consider a large population of individuals, each of whom possesses a particular pair of genes, of which each individual gene is classified as being of type A or type a. Assume that the proportions of individuals whose gene pairs are AA, aa, or Aa are, respectively, p_0, q_0, and r_0. Here, $p_0 + q_0 + r_0 = 1$. When two individuals mate, each contributes one of his or her genes, chosen at random, to the resultant offspring. Assuming that the mating occurs at random, in that each individual is equally likely to mate with any other individual, then 15 (CO4) (PO3)

i. Determine the proportions of individuals in the next generation whose genes are AA, aa, or Aa.

ii. Show that population follows the Hardy-Weinberg Law that is

iii. Let X_n denote the genetic state of her descendant in the nth generation, then define the transition probability matrix.

5. a) What is Markov chain and transition probability matrix? 7 (CO2) (PO2)

A particle performs a random walk with absorbing barriers, say, at 0 and 4. Whenever it is at any position $r (0 < r < 4)$, it moves to $r + 1$ with probability p or to $(r - 1)$ with probability q. Here, $p + q = 1$. Write down the transition probability matrix.

b) Let $\{X_n, n \geq 0\}$ be a Markov chain with three states 0, 1, 2 and with transition matrix: 8 (CO2) (PO2)

$$T = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

and the initial distribution $Pr\{X_0 = i\} = \frac{1}{3}, i = 0, 1, 2$. Compute the following probabilities

i. $Pr\{X_2 = 2, X_1 = 1, X_0 = 2\}$

ii. $Pr\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\}$

c) Define Poisson process. Derive the distribution of interarrival and waiting time. 10 (CO2) (PO2)

6. a) Suppose that each of three men at a party throws his hat into the center of the room. The hats are first mixed up and then each man randomly selects a hat. What is the probability that none of the three men selects his own hat? 7 (CO1) (PO1)

b) Suppose that the number of people who visit a yoga studio each day is a Poisson random variable with mean λ. Suppose further that each person who visits is, independently, female with probability p or male with probability $1 - p$. Find the joint probability that exactly n women and m men visit the academy today. 8 (CO1) (PO1)

c) An automobile insurance company classifies each of its policyholders as being of one of the types $i = 1, \ldots, k$. It supposes that the numbers of accidents that a type i policyholder has in successive years are independent Poisson random variables with mean $\lambda_i, i = 1, \ldots, k$. The probability that a newly insured policyholder is type i is p_i, $\Sigma_{i=1}^{k} p_i = 1$. Given that a policyholder had n accidents in her first year, what is the expected number that she has in her second year? What is the conditional probability that she has m accidents in her second year? 10 (CO1) (PO1)