8

ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

MID SEMESTER EXAMINATION SUMMER SEMESTER, 2022-2023
DURATION: 1 HOUR 30 MINUTES FULL MARKS: 75

Math 4441: Probability and Statistics

Programmable calculators are not allowed. Do not write anything on the question paper.
Answer **all 3 (three)** questions. Figures in the right margin indicate full marks of questions with corresponding COs and POs in parentheses.

1. Suppose an urn has 5 blue balls and 8 red balls. Someone picks balls randomly from the urn with replacement where the replacement adds an extra ball of the same color. More specifically, each time a ball is drawn from the urn, it is returned to the urn along with an additional ball of the same color. Suppose three balls are drawn in this way. (CO1) (PO1)
   a) Calculate the probability that all the three balls are blue. 5
   b) Calculate the probability that only one ball, among the three balls, is blue. 7
   c) Calculate the probability that the first ball drawn is a blue ball given that the second ball is blue. Assume that the third ball has not been drawn yet. 10
   d) With the application of the conditional Bayes' Theorem, calculate the probability that the first ball drawn is a blue ball given that both the second and the third balls are blue. 8

2. Consider a random experiment named as pick-and-drop (P&D), where Mr. Rahim picks 5 balls one-after-another from an urn without replacement. Assume that there are 5 blue balls and 8 red balls in the urn. After drawing the 5 balls from the urn, Mr. Rahim observes the color of the balls, counts the number of blue and red balls; and then, returns the balls in the urn. (CO2) (PO2)
   a) Let X be the random variable that represents the number of times Mr. Rahim has to run the P&D experiment to get exactly 2 blue balls and 3 red balls in a run of the P&D experiment. Find the PMF X, and the probabilities $P[X = 3]$ and $P[X > 3]$. 10
   b) Assume three friends of Mr. Rahim also want to run the P&D experiment until each of them observes exactly 2 blue balls and 3 red balls in a run of the P&D. Let Y be the random variable that represents the number of times the P&D experiment is executed collectively by the three friends of Mr. Rahim. Find the PMF and the expected value of Y. 8
   c) Assume one of the friends of Mr. Rahim puts 11 green balls in the urn. Now, Mr. Rahim randomly draws 6 balls from the urn with replacement. Find the probability that he draws exactly 2 blue balls, 2 red balls and 2 green balls using their multivariate PMF. 7

3. The bivariate PDF of random variables X and Y is given by (CO1) (PO1)

$$f_{X,Y}(x,y) = \begin{cases} c, & \text{for } -1 \le x \le 1, |x| \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

   a) Find the value of c and the probability $P[-1 < X < 1, 0.4 < Y < 1.0]$. 6
   b) Find the bivariate CDF of X and Y, $F_{X,Y}(x,y)$, for the region $-1 \le x \le 1, |x| \le y \le 1$. 7
   c) Find the marginal PDF of X and Y. 7

Appendix - B
General Formulas

Name	Formula
Conditional Probability	$P[A\|B] = \frac{P[AB]}{P[B]}, P[B] > 0$
Product Rule	$P[AB] = P[A]P[B\|A]$ $P[A_1 \dots A_n] = P[A_1]P[A_2\|A_1] \dots P[A_n\|A_1 \dots A_{n-1}]$
Conditional Product Rule	$P[AB\|C] = P[A\|C]P[B\|AC]$
Sum Rule	$P[A] = \sum_{i=1}^{n} P[A\|B_i]P[B_i]$
Conditional Sum Rule	$P[A\|C] = \sum_{i=1}^{n} P[A\|B_i, C]P[B_i\|C]$
Bayes' Theorem	$P[B_i\|A] = \frac{P[A\|B_i]P[B_i]}{\sum_{i=1}^{n} P[A\|B_i]P[B_i]}$
Conditional Bayes' Theorem	$P[B_i\|A, C] = \frac{P[A\|B_i,C]P[B_i\|C]}{\sum_{i=1}^{n} P[A\|B_i,C]P[B_i\|C]}$
Independence of Events	$P[AB] = P[A]P[B]$
Expected Value	$E[X] = \sum_{x \in S_X} x P_X(x)$ $E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$
Variance	$Var[X] = E[(X - \mu_X)^2] = E[X^2] - (\mu_X)^2$
Marginal PMF	$P_X(x) = \sum_{x \in S_X} P_{XY}(x, y)$ $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy$
Covariance	$Cov[X, Y] = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y$
Correlation	$r_{XY} = E[XY]$
Correlation Coefficient	$\rho_{XY} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}}$

Appendix - A

PMF/PDF, expected values and variance of known Random Variables

Families of Distribution	PMF or PDF	Expectation	Variance
Bernoulli $X \sim Ber(p)$	$P_X(x) = \begin{cases} p^x(1-p)^{1-x}, & x = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$	p	$p(1-p)$
Geometric $X \sim Geom(p)$	$P_X(x) = \begin{cases} p(1-p)^{x-1}, & x = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$
Binomial $X \sim Bin(n, p)$	$P_X(x) = \begin{cases} \binom{n}{x} p^x(1-p)^{n-x}, & x = 0, 1, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$	np	$np(1-p)$
Pascal/-ve Binomial $X \sim pascal(k, p)$	$P_X(x) = \begin{cases} \binom{x-1}{k-1} p^k(1-p)^{x-k}, & x = k, k+1, \ldots \\ 0, & \text{otherwise.} \end{cases}$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$
Poisson $X \sim Poisson(\lambda)$	$P_X(x) = \begin{cases} \frac{(\lambda T)^x e^{-(\lambda T)}}{x!}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$	λT	λT
Hyper Geometric $X \sim HGem(r, g, n)$	$P_X(x) = \begin{cases} \frac{\binom{r}{x}\binom{g}{n-x}}{\binom{r+g}{n}}, & x = 0, 1, \ldots, \min(r, n) \\ 0, & \text{otherwise.} \end{cases}$	$\frac{rn}{r+g}$	$\frac{nrg}{(r+g)^2}\left(1-\frac{n-1}{r+g-1}\right)$
Uniform (discrete) $X \sim unif(a, b)$	$P_X(x) = \begin{cases} \frac{1}{b-a+1}, & x = a, a+1, \ldots, b \\ 0, & \text{otherwise.} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)(b-a+2)}{12}$
Exponential $X \sim exp(\lambda)$	$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gaussian $X \sim N(\mu, \sigma^2)$	$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma}}, & -\infty < x < +\infty \\ 0, & \text{otherwise.} \end{cases}$	μ	σ^2
Uniform (Continuous) $X \sim unif(a, b)$	$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Gamma $X \sim gam(r, \lambda)$	$f_X(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{\Gamma(r)}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$	$\frac{r}{\lambda}$	$\frac{r}{\lambda^2}$
Multinomial	$P_{X_1,\ldots,X_r}(x_1, \ldots, x_r) = \begin{cases} \binom{n}{x_1 \ldots x_r} p_1^{x_1} \ldots p_r^{x_r}, & x_i = 0, \ldots, n, 0 \leq i \leq r, \sum_{i=0}^{r} x_i = n \\ 0, & \text{otherwise.} \end{cases}$		
Multivariate Hyper geometric	$P_X(x) = \begin{cases} \frac{\binom{n_1}{x_1}\binom{n_2}{x_2} \ldots \binom{n_r}{x_r}}{\binom{n_1 + \cdots + n_r}{x_1 + \cdots + x_r}}, & x = 0, 1, \ldots, \min(r, n) \\ 0, & \text{otherwise.} \end{cases}$		