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ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

SEMESTER FINAL EXAMINATION **SUMMER SEMESTER, 2022-2023**
DURATION: 3 HOURS **FULL MARKS: 150**

Math 4641: Numerical Methods

Programmable calculators are not allowed. Do not write anything on the question paper.
Answer **all 6 (six)** questions. Figures in the right margin indicate full marks of questions with corresponding COs and POs in parentheses.

1. a) Define numerical methods. Briefly explain why we need them. 2 (CO3) (PO1)

   b) i. Compare and contrast the following root-finding numerical algorithms– Bisection method, Newton-Raphson method, Secant method, and False Position method. 3 (CO3) (PO1)

   ii. What are the drawbacks of the Bisection method? Explain with proper examples and illustrations. 6 (CO3) (PO1)

   c) Suppose, R is an arbitrary number. Prove that the Newton-Raphson formula for finding the square root of R, which is mathematically denoted as $\sqrt{R}$, is as shown in Equation 1. 5 (CO2) (PO1)

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{R}{x_i}\right) \tag{1}$$

   d) The *Golden Ratio*, often denoted by the Greek letter ϕ (phi), is the ratio of two quantities if their ratio is the same as the ratio of their sum to the larger of the two quantities. Expressed algebraically, for quantities a and b with $a > b > 0$, the equality $\frac{a+b}{a} = \frac{a}{b} = \phi$ holds true. The constant ϕ is equal to the positive root of a particular function $f(x)$. The numerical value of that root is,

$$x^{+}_{root} = \phi = \frac{1+\sqrt{5}}{2} = 1.618033988749\ldots$$

   i. From the given scenario, formulate the function $f(x)$. 3 (CO1) (PO2)

   ii. Apply the Secant method to estimate the value of the golden ratio ϕ. The initial guesses are $x_{-1} = 0.5$ and $x_0 = 1$. Demonstrating the step-by-step mathematical procedure, conduct 4 iterations, and find the relative approximate error ($|\epsilon_a|\%$) and the number of significant digits that are at least correct (m) at the end of each iteration. Draw Table 1 on your answer script and fill it out after performing the necessary calculations. 11 (CO2) (PO1)

Table 1: The relevant values obtained in the answer of Question 1.d)ii

Iteration	x_{i-1}	x_i	x_{i+1}	$\lvert\epsilon_a\rvert\%$	m	$f(x_{i+1})$
1	0.5	1				
2						
3						
4						

2. a) Define Round-off Error. What are the different ways through which Round-off Error can be introduced? Explain with proper examples. $1+4$ (CO3) (PO1)

b) One of the fundamental relationships between the sine and cosine functions is given by the Pythagorean trigonometric identity, which is, 14 (CO2) (PO1)

$$\sin^2(x) + \cos^2(x) = 1 \quad (2)$$

One of the many proofs of this identity involves utilizing the Taylor expansions of its constituent transcendental functions.

The Taylor series for a function $f(x)$ is —

$$f(x+h) = f(x) + f'(x)\frac{h}{1!} + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4!} + f'''''(x)\frac{h^5}{5!} + \dots$$

Using the Taylor series (with at least the first 5 terms), derive the Maclaurin series of —

i. $\sin^2(x)$

ii. $\cos^2(x)$

Then use these two series to prove the Pythagorean trigonometric identity, as delineated in Equation 2.

c) The Maclaurin series of the exponential function e^x is —

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

i. Why is the exponential function e^x categorized as a transcendental function? 1 (CO4) (PO1)

ii. Using the Remainder Theorem, establish the bounds of the truncation error in the representation of $e^{1.69}$ if only the first 5 terms of the series are used. 5 (CO2) (PO1)

3. a) Prove that a polynomial of degree n or less that passes through $n+1$ data points is unique. 6 (CO4) (PO1)

b) Why do we use polynomials as interpolants? Briefly describe the properties of polynomials for which they are used as interpolants in numerical analysis. 6 (CO3) (PO1)

c) The specific heat capacity C of a substance is defined as the amount of heat that is required to raise the temperature of unit mass of that particular substance by 1 degree. Suppose, you are conducting an experiment to determine how much heat is required to bring some volume of water to its boiling point. The values of specific heat capacity C of water that you have calculated at different temperature values T are shown in Table 2. 8 (CO2) (PO1)

Table 2: Specific heat C of water as a function of temperature T for Question 3.c

Temperature, T (°C)	Specific Heat, C ($Jkg^{-1}{}^{\circ}C^{-1}$)
22	4181
42	4179
52	4186
82	4199
100	4217

Determine the value of the specific heat at $T = 61°C$ using the Newton's Divided Difference method of interpolation and a second order polynomial.

4. a) The general linear regression model for predicting the response for a given set of n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is $7+5$ (CO4) (PO1)

$$y = a_0 + a_1 x$$

where a_0 and a_1 are the two parameters of the regression model.

i. Derive the formulae for finding the optimal values of a_0 and a_1.

ii. Prove that the values of a_0 and a_1 obtained using the formulae derived in Question 4.a)i correspond to the absolute minimum of the optimization criterion used.

b) What are the types of processes that can be represented using the Exponential model of nonlinear regression? Briefly explain with suitable real-world examples. 5 (CO3) (PO1)

c) Suppose, you are given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ and you want to regress them to the Exponential model $y = e^{bx}$. 5 (CO3) (PO1)

- Without performing any data transformation, you obtain the regression model $y = e^{k_1 x}$.
- With data transformation via linearization, you obtain the regression model $y = e^{k_2 x}$.

With proper reasoning, explain whether k_1 and k_2 are equal or not.

5. a) i. Using the method of coefficients, derive the formula for the Single-segment Trapezoidal rule. 7 (CO4) (PO1)

ii. Applying the formula from Question 5.a)i, derive the formula for the Multi-segment Trapezoidal rule. 6 (CO4) (PO1)

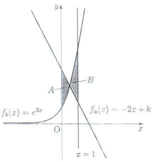

Figure 1: The shaded regions A and B for Question 5.b.

b) In Figure 1, the area of the shaded regions A and B, portrayed beneath the two functions $f_a(x)$ and $f_b(x)$ respectively, are equal. The aforementioned functions are, $8+6$ (CO2) (PO1)

$$f_a(x) = -2x + k$$
$$f_b(x) = e^{2x}$$

i. Using the Composite Simpson's 3/8 Rule, with $n = 9$ segments, approximate the value of k.

ii. Analytically determine the exact value of k, denoted by k_{exact}, and use it to calculate the true error (E_t) and the absolute relative true error $(|\epsilon_t|\%)$.

6. a) Observe the lightly shaded solid object portrayed in Figure 2.

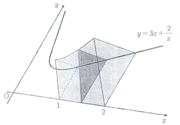

Figure 2: The solid geometric object for Question 6.a.

Its base is enclosed by a curve $y = 3x + \frac{2}{x}$ (for $x > 0$), the x-axis, the straight line $x = 1$, and the straight line $x = 2$. When this solid object is cut by a plane perpendicular to the x-axis and the xy-plane, the resulting cross-section is an equilateral triangle (as indicated by the darkly shaded cross-sectional area).

i. Formulate an integrand function $g(x)$ whose integral value within the limits $[1, 2]$ will yield the volume of the solid object. 5 (CO1) (PO2)

ii. Approximate the volume of the solid object, which is represented by $I = \int_1^2 g(x)dx$, using Euler's method with a step size of $h = 0.5$. 8 (CO2) (PO1)

b) i. Find the eigenvalues and associated eigenvectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$ 6 (CO2) (PO1)

ii. Determine the order and degree of each of these higher-order differential equations 2 (CO4) (PO1)

(a) $x^2 \frac{dy}{dx} - \left(\frac{d^3y}{dx^3}\right)^3 + 69 = 0$

(b) $\left(\frac{dy}{dx}\right)^2 - 420xy = 3$

(c) $\frac{d^3y}{dx^3} + x^4 \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^4 = \sin x$

(d) $\frac{dy}{dx} + xy = x^2$

c) Suppose, $u = u(x, y)$ is a function of two variables that we only know at discrete grid points (x_i, y_j) given in the matrix 5 (CO2) (PO1)

$$[u_{i,j}] = \begin{bmatrix} 5.1 & 6.5 & 7.5 & 8.1 & 8.4 \\ 5.5 & 6.8 & 7.8 & 8.3 & 8.9 \\ 5.5 & 6.9 & 9.0 & 8.4 & 9.1 \\ 5.4 & 9.6 & 9.1 & 8.6 & 9.4 \end{bmatrix}$$

Find the approximate value of the following partial derivatives

(a) $u_x(x_2, y_4)$ (b) $u_y(x_2, y_4)$ (c) $u_{xx}(x_2, y_4)$

(d) $u_{yy}(x_2, y_4)$ (e) $u_{xy}(x_2, y_4)$

using the central difference formula. Consider $h = 0.5$ and $k = 0.2$.
Note: Assume 1-based indexing.