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ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)
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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

Semester Final Examination | Summer Semester, A. Y. 2022-2023
Course No.: EEE 4407 | Time: 3 Hours
Course Title: Random Signals and Processes | Full Marks: 150

There are **6 (six)** questions. Answer **all the** questions. All questions carry equal marks. Marks in the margin indicate full marks. Programmable calculators are not allowed. Do not write on this question paper. Symbols and acronyms have their usual meanings.

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1. a) $\mathbf{X}$ is the 3-dimensional Gaussian random vector with expected value $\boldsymbol{\mu}_{\mathbf{X}} = [4 \quad 8 \quad 6]'$ and covariance 13 (CO3 PO2)

$$\mathbf{C_X} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}$$

Calculate
i) The correlation matrix, $\mathbf{R_X}$
ii) The PDF of the first two components of $\mathbf{X}$, $f_{X_1,X_2}(x_1, x_2)$
iii) The probability that $X_1 > 8$

b) Given the Gaussian random vector $\mathbf{X}$ in question 1(a) above, $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$, where, 12 (CO3 PO2)

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix}$$

and $\boldsymbol{b} = [-4 \quad -4]'$

Calculate
i) The expected value, $\boldsymbol{\mu}_{\mathbf{Y}}$
ii) The covariance, $\mathbf{C_Y}$
iii) The correlation, $\mathbf{R_Y}$
iv) The probability that $\mathbf{-1 \leq} \boldsymbol{Y_2} \mathbf{\leq 1}$

2. a) Deer ticks can carry both Lyme disease and human granulocytic ehrlichiosis (HGE). In a study of ticks in the Midwest, it was found that 16% carried Lyme disease, 10% had HGE, and that 10% of the ticks that had either Lyme disease or HGE carried both diseases. 10 (CO1 PO1)
   i. Find the probability P[LH] that a tick carries both Lyme disease (L) and HGE(H).
   ii. Find the conditional probability that a tick has HGE given that it has Lyme disease.

b) Monitor a phone call where classify the call as a voice call (V) if someone is speaking, or a data call (D) if the call is carrying a modem or fax signal. Classify the call as long (L) if the call lasts for more than three minutes; otherwise classify the call as brief (B). Based on data collected by the telephone company, we use the following probability model: P[V]= 0.7, P[L]= 0.6, P[VL]= 0.35. Find the following probabilities: 9 (CO1 PO1)
i. P [DL], ii. P [D ∪ L],
iii. P [VB], iv. P [V ∪ L],
v. P [V ∪ D], vi. P [LB].

c) In a cellular phone system, a mobile phone must be paged to receive a phone call. However, paging attempts do not always succeed because the mobile phone may not receive the paging signal clearly. Consequently, the system will page a phone up to three times before giving up. If a single paging attempt succeeds with probability 0.8, sketch a probability tree for this experiment and find the probability P[F] that the phone is found. (CO1 PO1)

3. a) A radio station gives a pair of concert tickets to sixth caller who knows the birthday of the performer. For each person who calls, the probability is 0.75 of knowing the performer's birthday. All calls are independent. 13 (CO2 PO2)

i. Find the PMF of L, the number of calls necessary to find the winner.
ii. Find the probability of finding the winner on the tenth call.
iii. Find the probability that the station will need nine or more calls to find a winner.

b) X is a continuous uniform $(-5, 5)$ random variable. 12 (CO2 PO2)

i) Find PDF $f_X(x)$.
ii) Find CDF $F_X(x)$.
iii) Find $E[X]$.
iv) Find $E[X^5]$.
v) Find $E[e^X]$.

4. a) Find joint probability mass function and marginal probability. For a constant $a > 0$, random variables X and Y have joint PDF 13 (CO2 PO2)

$$f_{X,Y}(x,y) = \begin{cases} 1/a^2 & 0 \le x \le a, \quad 0 \le y \le a, \\ 0 & otherwise. \end{cases}$$

Find the CDF and PDF of random variable

$$W = \max\left(\frac{X}{Y}, \frac{Y}{X}\right).$$

b) Random variables X and Y have joint PDF 12 (CO2 PO2)

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{5x^2}{2} & -1 \le x \le 1, \quad 0 \le y \le x^2, \\ 0 & otherwise. \end{cases}$$

i. Find E[X] and Var[X].
ii. Find E[Y] and Var[Y].
iii. Find Cov[X, Y].
iv. Find E[X + Y].
v. Find Var[X + Y].

5. a) J and K are independent random variables with probability mass functions 13 (CO3 PO2)

$$P_J(j) = \begin{cases} 0.2 & j = 1, \\ 0.6 & j = 2, \\ 0.2 & j = 3, \\ 0 & otherwise, \end{cases} \qquad P_K(k) = \begin{cases} 0.5 & k = -1, \\ 0.5 & k = 1, \\ 0 & otherwise. \end{cases}$$

Find the MGF of $M = J + K$. Also find $E[M^3]$, $E[M^4]$ and $P_M(m)$.

b) In a production line for 1000 Ω resistors, the actual resistance in ohms of each resistor is a uniform (950, 1050) random variable R. The resistances of different resistors are independent. The resistor company has an order for 1% resistors with a resistance between 990 Ω and 1010 Ω. An automatic tester takes one resistor per second and measures its exact resistance. (This test takes one second). The random process N(t) denotes the number of 1% resistors found in t seconds. The random variable T_r seconds is the elapsed time at which r 1% resistors are found. 12 (CO5 PO2)

i. Find p, the probability that any single resistor is a 1% resistor.
ii. Find the PMF of N(t).
iii. Find E[T1] seconds, the expected time to find the first 1% resistor.
iv. Find the probability that the first 1% resistor is found in exactly 5 seconds.
v. If the automatic tester finds the first 1% resistor in 10 seconds, Find E[T2|T1=10], the conditional expected value of the time of finding the second 1% resistor?

6. a) Consider an experiment that produces a Bernoulli random variable with probability of success q. In order to estimate q, we perform the experiment that produces this random variable n. In this experiment, q is a sample value of a random variable, Q, with PDF 13 (CO4 PO2)

$$f_Q(q) = \begin{cases} 6q(1-q) & 0 \leq q \leq 1 \\ 0 & otherwise \end{cases}$$

In Appendix A, we can identify Q as a beta ($i = 2, j = 2$) random variable. To estimate Q we perform n independent trials of the Bernoulli experiment. The number of successes in the n trials is a random variable K. Given an observation $K = k$, evaluate the following estimates of Q:
i. The blind estimate $\hat{q}_B$,
ii. The maximum likelihood estimate $\hat{q}_{ML}(k)$,
iii. The maximum a posteriori probability estimate $\hat{q}_{MAP}(k)$.

b) A telemetry voltage V, transmitted from a position sensor on a ship's rudder, is a random variable with PDF 12 (CO4 PO2)

$$f_V(v) = \begin{cases} 1/12 & -6 \leq v \leq 6, \\ 0 & \text{otherwise.} \end{cases}$$

A receiver in the ship's control room receives $R = V + X$, The random variable X is a Gaussian $(0, \sqrt{3})$ noise voltage that is independent of V. The receiver uses R to calculate a linear estimate of the telemetry voltage:

$$\hat{V} = aR + b$$

Find

i. the expected received voltage E[R],
ii. the variance Var[R] of the received voltage,
iii. the covariance Cov[V, R] of the transmitted and received voltages,
iv. a* and b*, the optimum coefficients in the linear estimate,
v. e*, the minimum mean square error of the estimate.

A.2 Continuous Random Variables

Beta (i, j)

For positive integers i and j, the beta function is defined as

$$\beta(i, j) = \frac{(i + j - 1)!}{(i - 1)!(j - 1)!}$$

For a $\beta(i, j)$ random variable X,

$$f_X(x) = \begin{cases} \beta(i, j)x^{i-1}(1 - x)^{j-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{i}{i + j}$$

$$\text{Var}[X] = \frac{ij}{(i + j)^2(i + j + 1)}$$

Appendix B

Table for The standard normal CDF

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

The standard normal CDF $\Phi(y)$.